Breaking supersymmetry in a one-dimensional random Hamiltonian

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Breaking supersymmetry in a one-dimensional random Hamiltonian

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Abstract
The one-dimensional supersymmetric random Hamiltonian $H_{\text{susy}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi^2 + \phi'$, where $\phi(x)$ is a Gaussian white noise of zero mean and variance g, presents particular spectral and localization properties at low energy: a Dyson singularity in the integrated density of states (IDoS) $N(E) \sim 1/\ln^2 E$ and a delocalization transition related to the behavior of the Lyapunov exponent (inverse localization length) vanishing like $\gamma(E) \sim 1/|\ln E|$ as $E \to 0$. We study how this picture is affected by breaking supersymmetry with a scalar random potential: $H = H_{\text{susy}} + V(x)$, where $V(x)$ is a Gaussian white noise of variance σ. In the limit $\sigma \ll g^3$, a fraction of states $N(0) \sim g/\ln^2(g^3/\sigma)$ migrate to the negative spectrum and the Lyapunov exponent reaches a finite value $\gamma(0) \sim g/\ln(g^3/\sigma)$ at $E = 0$. The exponential (Lifshits) tail of the IDoS for $E \to -\infty$ is studied in detail and is shown to involve a competition between the two noises $\phi(x)$ and $V(x)$, whatever the larger is. This analysis relies on analytic results for $N(E)$ and $\gamma(E)$ obtained by two different methods: a stochastic method and the replica method. The problem of extreme value statistics of eigenvalues is also considered (distribution of the nth excited-state energy). The results are analyzed in the context of classical diffusion in a random force field in the presence of random annihilation/creation local rates.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The study of spectral and localization properties of one-dimensional (1D) random Hamiltonians has stimulated a huge activity since the pioneering works of Dyson [1], Schmidt

[2], Frisch and Lloyd [3], Lifshits [4] and many others (one may find references to most of the important works in the review [61] and the books [5, 7]). Because the dimension plays an important role in localization problems [6], the strictly one-dimensional situation misses some features of the higher dimension case (like a weak localization regime). On the other hand, the one-dimensional case allows us to make use of powerful nonperturbative methods and study subtle properties which are much more difficult to tackle in higher dimensions. The random Schrödinger Hamiltonians $H_{\text{scalar}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, where $V(x)$ is a random function, have been studied in great detail [5] and their properties are rather generic under the assumption that $V(x)$ is correlated on a small length scale and $\int \mathrm{d}x \langle V(x)V(0)\rangle$ remains finite[4]: exponential tail in the density of states[5] at low energies (Lifshits singularity) [2, 3, 5, 9, 11–13] and decreasing Lyapunov exponent (inverse localization length) at high energy [5, 13] $\gamma \propto 1/E$ for $E \rightarrow +\infty$. The situation can be quite different if the Hamiltonian possesses some symmetry preserved by the introduction of the random potential. Such a situation occurs in the case of the supersymmetric random Hamiltonian

$$H_{\text{susy}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x). \tag{1}$$

This Hamiltonian has a positive spectrum, a direct consequence of the fact that it can be factorized in the form $H_{\text{susy}} = Q^\dagger Q$ with $Q = -\frac{\mathrm{d}}{\mathrm{d}x} + \phi(x)$ and $Q^\dagger = \frac{\mathrm{d}}{\mathrm{d}x} + \phi(x)$. Moreover, it is worth pointing out that $H_{\text{susy}} \equiv H_+ = Q^\dagger Q$ and its supersymmetric partner $H_- = QQ^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi^2 - \phi'$ are the two components of the square of the Dirac Hamiltonian $\mathcal{H}_D = \sigma_2 \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}x} + \sigma_1 \phi(x)$, where σ_i are Pauli matrices: $\mathcal{H}_D^2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi^2 + \sigma_3 \phi'$. Therefore the Hamiltonian (1) arises naturally when studying random Dirac Hamiltonians. Besides its own interest for the physics of localization, this model is relevant in several physical contexts like classical diffusion in a random force field (Sinai problem) [14–18] (see section 1.1), organic conductors [19, 16] or spin chains (the spectrum of excitations of an antiferromagnetic spin-chain is linear at small energies like in the free fermion model; the precise mapping of AF spin-chain to free fermions can be achieved thanks to a Jordan–Wigner transformation) [17, 20–24]; see the review provided in [15]. The relation to discrete models has been discussed: the supersymmetric Hamiltonian is the continuum limit of a discrete tight-binding Hamiltonian with off-diagonal disorder [16]. It is also the continuum limit of a tight-binding Hamiltonian with diagonal disorder at the band center [25, 5] (this point has been recently rediscussed in [26]). The supersymmetry is responsible for rather particular spectral and localization properties. For the sake of concreteness, let us choose for $\phi(x)$ a Gaussian white noise of zero mean, $\langle \phi(x) \rangle = 0$ and $\langle \phi(x)\phi(x') \rangle = g\delta(x - x')$. In the low-energy limit, $E \ll g^2$, the integrated density of states (IDoS) presents the Dyson singularity $N(E) \simeq 2g/\ln^2(g^2/E)$ [14, 15, 20, 25], similar to the one of the spring chain with random masses [1]. The Lyapunov exponent vanishes as $\gamma(E) \simeq 2g/\ln(g^2/E)$ [15, 20], indicating a delocalization transition. This delocalization transition is supported by studying other quantities: (i) the statistical properties of the zero mode wavefunction [27–29] indicate long-range power-law correlations (like the Lyapunov exponent analysis, these calculations do not account for boundary conditions). (ii) The distribution of the transmission probability through

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[4] Some interesting results have also been obtained in [7] in a situation where the correlation function grows at large distance like $\langle V(x)V(0) \rangle \sim |x|^\eta$ with $\eta > 0$ (the case $\eta = 1$ corresponds to a Brownian motion).

[5] The form of the exponential Lifshits tail depends on the details of the distribution of the random potential. Note that the spectrum of the Hamiltonian $H_{\text{scalar}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ can also present power-law singularity: for a random potential describing a weak concentration of impurities of negative weights, each trapping a localized state at energy $E_0 < 0$, the spectrum presents a power-law singularity near E_0, with an exponent proportional to the concentration of impurities; such a singularity is called a Halperin singularity [2, 3, 8–10]).

a finite slab of length L at zero energy. In particular the average transmission decreases like $1/\sqrt{L}$ [30], that is slower than the behavior $1/L$ for a quasi-1D conducting weakly disordered wire. (iii) Time-delay distribution presents a log–normal distribution at zero energy [30, 31]. (iv) The conductivity is found to be finite at $E = 0$ [32]. (v) Finally, the study of extreme value statistics of energy levels indicates spectral correlations for $E \rightarrow 0$ [33]. In the high-energy limit $E \rightarrow \infty$, the localization properties are quite unusual since the Lyapunov exponent does not vanish but reaches a finite value $\gamma(E \rightarrow \infty) \simeq g/2$. This property is due to the singular nature of the potential $\phi^2 + \phi'$ with ϕ a white noise. When the potential is regularized by introducing a small but finite correlation length, it has been shown in [16, 34] that the Lyapunov exponent decreases as $\gamma \propto 1/E$ for largest energies, as for the random Hamiltonian $H_{\text{scalar}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$. If the random function $\phi(x)$ possesses a finite mean value $\langle \phi(x) \rangle = \mu g$, logarithmic singularities are converted into power-law singularities [15, 25]. Extension to more general situations has been considered in [35], where spectrum and localization have been studied for the most general random Dirac 1D Hamiltonian (random mass, random scalar field and random gauge field); however, such a study still preserves the (particle–hole) symmetry of the Hamiltonian (note that the distribution of the local DoS for this model has been investigated in [36]).

The aim of the present paper is to discuss the effect of the addition of a scalar random potential that breaks the supersymmetry,

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x) + V(x).$$ (2)

We will mostly consider the case when the functions ϕ and V are two uncorrelated Gaussian white noises with variances $\langle \phi(x)\phi(x') \rangle = g\delta(x - x')$ and $\langle V(x)V(x') \rangle = \sigma\delta(x - x')$. The case with a finite $\langle \phi(x) \rangle$ will be studied in section 4 with the replica method. The case of correlated Gaussian white noises ϕ and V will be discussed in appendix A, where it is mapped onto the problem of uncorrelated noises. Our purpose is to study how the spectral and localization properties of H_{susy} are modified when introducing the scalar potential. A first obvious change is that the spectrum of H is not restricted to be positive. Natural questions are therefore: what is the number of states sent to $\mathbb{R}^-$ by the introduction of the potential $V(x)$, how their energies are distributed? How the delocalization at $E \rightarrow 0$ for the Hamiltonian H_{susy} is affected?

The paper is organized as follows. After giving a physical motivation for our model right hereafter, we study the spectral and localization properties of H in sections 2 and 3 respectively. Our approach relies on well-established techniques of stochastic differential equations. In section 4, we employ the replica method in order to find other analytical expressions for the IDoS and the Lyapunov exponent and consider the more general case of a finite $\langle \phi(x) \rangle$.

1.1. A motivation: branching random walks in a disordered environment

Let us first recall the well-known relation between the Fokker–Planck equation (FPE) describing classical diffusion in a force field $\phi(x)$ and the Schrödinger equation for a potential $\phi^2 + \phi'$. Let us consider the Langevin equation $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 2\phi(x(t)) + \sqrt{2}\eta(t)$, where the Langevin force $\eta(t)$ is a normalized white noise. This equation is related to the FPE $\partial_t P(x;t) = F_x P(x;t)$, where the forward generator reads $F_x = \partial_x^2 - 2\partial_x\phi(x)$. The FPE can be transformed into the Schrödinger equation $-\partial_t \psi(x;t) = H_{\text{susy}}\psi(x;t)$ thanks to the nonunitary transformation $P(x;t) = \psi_0(x)\psi(x;t)$ since

$$\psi_0(x)^{-1} F_x \psi_0(x) = -H_{\text{susy}} \qquad \text{where} \qquad \psi_0(x) = \mathrm{e}^{\int^x \mathrm{d}x' \phi(x')}.$$ (3)

Note that the operator transformation $F_x \to H_{\text{susy}}$ is isospectral. $\psi_0(x)$ is annihilated by the operator Q defined above: $Q\psi_0 = 0$. For a confining force field, $\psi_0(x)$ is the normalizable zero mode of H_{susy} and is related to the stationary distribution of the FPE: $P(x; t \to \infty) \simeq \psi_0(x)^2$.

In order to propose the physical interpretation of the last term of (2), we start from a discrete formulation of the problem of diffusion-controlled reaction in a one-dimensional quenched random potential landscape $\mathcal{V}_k$. Let us consider non-interacting particles on an infinite one-dimensional lattice with lattice spacing a. We label lattice sites by $k \in \mathbb{Z}$, corresponding to a position ka. We allow the local occupation number n_k for site k to take arbitrary positive integer values (bosonic particles). The transition rates between neighboring sites k and $k+1$ can be obtained from the Arrhenius law,

$$t_{k+1,k} = \frac{1}{a^2}\, \mathrm{e}^{\mathcal{V}_k - \mathcal{V}_{k+1}}, \tag{4}$$

where $\mathcal{V}_k$ is the potential at site k. The prefactor is chosen in order to obtain a well-defined continuum limit $a \to 0^+$. Additionally we consider the following chemical reactions: we allow particle replication $\mathrm{A} \to m\mathrm{A}, m \geqslant 2$, with a local rate $\beta_{m,k}$ and particle annihilation $\mathrm{A} \to \emptyset$ with a local rate γ_k. The reaction rates are supposed to be random quantities. Therefore, the model describes branching random walks in a one-dimensional disordered environment, including particle annihilation.

Let us study the particle distribution on the lattice: we denote n_k the occupation of site k. Its mean value obeys the following master equation:

$$\frac{\mathrm{d}\bar{n}_k}{\mathrm{d}t} = t_{k,k+1}\bar{n}_{k+1} + t_{k,k-1}\bar{n}_{k-1} - (t_{k+1,k} + t_{k-1,k})\bar{n}_k + (\beta_k - \gamma_k)\bar{n}_k, \tag{5}$$

where averaging $\overline{\cdots}$ is taken with respect to the random dynamics defined by rates (4) (not to be confused with averaging $\langle\cdots\rangle$ with respect to the quenched random potential $\mathcal{V}_k$ and random annihilation/creation rates). We have introduced $\beta_k = \sum_{m=1}^{\infty} m\beta_{m+1,k}$. For the continuum limit, we introduce the density $n(x = ka, t) = \bar{n}_k/a$. As $a \to 0$ we develop $\frac{1}{a}\bar{n}_{k\pm1} = n(x,t) \pm a\partial_x n(x,t) + \frac{1}{2}a^2\partial_x^2 n(x,t) + \cdots$. Moreover, we introduce the force field $\phi(x)$ via $\mathcal{V}_k - \mathcal{V}_{k+1} = a\phi(x = ka) + \frac{1}{2}a^2\phi'(x = ka) + \cdots$ what allows us to develop the transition rates (4) as

$$\begin{aligned}
t_{k,k\pm1} &= \frac{1}{a^2} \mp \frac{\phi(x)}{a} - \frac{\phi'(x)}{2} + \frac{\phi(x)^2}{2} + \cdots, \\
t_{k\pm1,k} &= \frac{1}{a^2} \pm \frac{\phi(x)}{a} + \frac{\phi'(x)}{2} + \frac{\phi(x)^2}{2} + \cdots.
\end{aligned} \tag{6}$$

We also introduce the notation $\gamma_k - \beta_k = V(x = ka)$ for the difference of annihilation rates and creation rates ($V(x) > 0$ corresponds to annihilation and $V(x) < 0$ to creation). The development yields the partial differential equation

$$\frac{\partial n(x,t)}{\partial t} = \frac{\partial^2 n(x,t)}{\partial x^2} - 2\frac{\partial}{\partial x}[\phi(x)n(x,t)] - V(x)n(x,t) = -H_{\text{FP}}n(x,t) \tag{7}$$

for the average particle density, with $H_{\text{FP}} = -F_x + V(x)$. We will consider the case where the random force field $\phi(x)$ and the random annihilation/creation rates $V(x)$ are correlated over small scale. For the large scale properties of the diffusion, the minimal model corresponds to assume that $\phi(x)$ and $V(x)$ are two Gaussian white noises. The mean value $\langle\phi(x)\rangle$ corresponds to the average drift of particles and $\langle V(x)\rangle$ is related to the average rate of particle annihilation at x. We will first consider the case $\langle\phi(x)\rangle = 0$ (the case of finite drift will be discussed in section 4). A finite average creation rate $\langle V(x)\rangle$ corresponds to a trivial global shift of the spectrum of H, therefore we will set $\langle V(x)\rangle = 0$.

We have introduced a Fokker–Planck-like differential operator H_{FP} which, as explained above, may be related to the Schrödinger operator (2) thanks to the isospectral transformation (3): $\psi(x, t) = \psi_0(x)n(x, t)$. Hence, the spectrum of H is of great interest for the diffusion problem. In particular, if we wish to determine the density $n(x, t|y, 0)$ with the initial condition $n(x, 0|y, 0) = \delta(x - y)$ we may rewrite in terms of the spectrum $\{E_\alpha, \Psi_\alpha(x)\}$ of H,

$$n(x, t|y, 0) = \frac{\psi_0(x)}{\psi_0(y)} \sum_\alpha \Psi_\alpha(x)\Psi_\alpha(y) \, \mathrm{e}^{-E_\alpha t}, \tag{8}$$

where $\psi_0(x)$ is the zero mode of H_{susy} given above. A first quantity to consider is the average occupation at x at time t after release of a particle at $y = x$ at time $t = 0$. We can use the translation invariance of the problem to identify the position average with averaging with respect to disorder,

$$\langle n(x, t|x, 0)\rangle = \lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{+L/2} \mathrm{d}x \, n(x, t|x, 0) = \int_{-\infty}^{+\infty} \mathrm{d}E \, \rho(E) \, \mathrm{e}^{-Et}, \tag{9}$$

where $\rho(E)$ denotes the density of states of H (we have omitted averaging in the rhs thanks to the self-averaging properties of the density of states). This relation shows that the low-energy properties of the quantum Hamiltonian are related to large time asymptotics for the return probability of the classical diffusion problem.

2. Spectral properties

In this section, we recall the phase formalism, the continuous version of the well-known Dyson–Schmidt method [1, 2, 37]. A clear presentation can be found in [5, 13]. The basic idea relates on the Sturm–Liouville theorem stating that the number of nodes of the one-dimensional wavefunction of energy E is equal to the number of normalizable states below E. The starting point is to convert the Sturm–Liouville problem[6] into a Cauchy problem[7] and study the statistical properties of the solution of this latter problem. The next step consists to separate the solution into an oscillating part and an envelope $\psi(x; E) = \rho_E(x) \sin\theta_E(x)$. The study of the phase $\theta_E(x)$ permits to analyze the spectral properties of the Hamiltonian H since it allows us to count the number of nodes of the wavefunction. The damping of the envelope characterizes its localization properties. Strictly speaking, $\psi(x; E)$ is the wavefunction only if E coincides with an eigenvalue $\psi(x; E_n) \propto \varphi_n(x)$, that is when the second boundary condition is satisfied $\psi(x = L; E_n) = 0$.

2.1. Ricatti variable

It is convenient to start by introducing the 'Ricatti' variable $z = \psi'/\psi - \phi$, the Schrödinger equation $H\psi = E\psi$ leads to the stochastic differential equation (SDE),

$$\frac{\mathrm{d}}{\mathrm{d}x}z(x) = -E - z(x)^2 - 2z(x)\phi(x) + V(x) \qquad \text{(Stratonovich).} \tag{10}$$

Since the random functions ϕ and V are understood to be the white noise limits of some physical regular noises (correlated over a finite length scale), the SDE must be understood in

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[6] A spectral problem is formulated as: find the solutions of $H\varphi(x) = E\varphi(x)$ for some boundary conditions, e.g. $\varphi(0) = \varphi(L) = 0$. On a finite interval, such solutions $(\varphi_n(x), E_n)$ exist only for discrete values of the energy $E \in \mathrm{Spec}(H) = \{E_n\}$.

[7] Solve $H\psi(x; E) = E\psi(x; E)$ for given initial conditions, e.g. $\psi(0; E) = 0$ and $\psi'(0; E) = 1$. Solutions exist $\forall E$.

the Stratonovich sense [38]. Relation (B.1) derived in appendix B allows us to simplify (10) in order to deal with one noise only

$$\mathrm{d}z \overset{\text{(law)}}{=} -(E + z^2)\,\mathrm{d}x + \sqrt{\sigma + 4gz^2}\,\mathrm{d}W(x) \qquad \text{(Stratonovich)}, \qquad (11)$$

where $W(x)$ is a normalized Wiener process (primitive of a white noise). We define $\beta(z) = \sqrt{\sigma + 4gz^2}$. This Langevin equation is related to a Fokker–Planck equation (FPE) $\partial_x T(z; x) = F_z T(z; x)$ where $F_z = \partial_z(E + z^2) + \frac{1}{2}[\partial_z\beta(z)]^2$ is the forward generator. This equation admits a stationary solution for a constant flow. The current of z through $\mathbb{R}$ corresponds to the number of divergencies of the Ricatti variable per unit length, therefore to the number of zeros of the wavefunction per unit length. This is precisely the average integrated density of states (IDoS) per unit length $N(E)$. Therefore

$$N(E) = (z^2 + E)T(z) + \frac{1}{2}\beta(z)\frac{\mathrm{d}}{\mathrm{d}z}[\beta(z)T(z)]. \qquad (12)$$

We recover on this particular case the general Rice formula $\lim_{z\to\infty} z^2 T(z) = N(E)$. We introduce the function $\mathcal{U}(z) = 4g\int_0^z \mathrm{d}z' \frac{E + z'^2}{\beta(z')^2}$,

$$\mathcal{U}(z) = z + \sqrt{\frac{\sigma}{4g}}\left(\frac{4Eg}{\sigma} - 1\right)\arctan\left(\sqrt{\frac{4g}{\sigma}}\,z\right), \qquad (13)$$

we obtain the distribution

$$T(z) = \frac{2N(E)}{\beta(z)}\,\mathrm{e}^{-\frac{1}{2g}\mathcal{U}(z)}\int_{-\infty}^z \frac{\mathrm{d}z'}{\beta(z')}\,\mathrm{e}^{\frac{1}{2g}\mathcal{U}(z')}. \qquad (14)$$

Imposing normalization gives an explicit expression of the IDoS.

2.2. Phase and envelope

The phase formalism introduces another set of variables that give a more transparent picture to analyze spectrum and localization.

Positive part of the spectrum: $E = +k^2$. We write $H_{\text{susy}} = Q^\dagger Q$ with $Q = -\frac{\mathrm{d}}{\mathrm{d}x} + \phi(x)$ and $Q^\dagger = \frac{\mathrm{d}}{\mathrm{d}x} + \phi(x)$. $H\psi = E\psi$ with $E = k^2$ can be cast in the form

$$Q\psi = k\chi \qquad (15)$$

$$Q^\dagger\chi = \left(k - \frac{1}{k}V(x)\right)\psi. \qquad (16)$$

We introduce phase θ and envelope e^ξ variables,

$$\psi(x) = \mathrm{e}^{\xi(x)}\sin\theta(x) \qquad (17)$$

$$\chi(x) = -\mathrm{e}^{\xi(x)}\cos\theta(x) \qquad (18)$$

with the initial conditions $\theta(0) = 0$ and $\xi(0) = 0$. The phase is related to the Ricatti variable by $z = -\frac{Q\psi}{\psi} = k\,\mathrm{cotg}\,\theta$. The interest to deal with this couple of variables lies in the basic idea of the phase formalism, i.e. the node counting method: the IDoS coincides with the number of nodes of the wavefunction that can be obtained from the evolution of the cumulative phase. The Lyapunov exponent (inverse localization length) is defined as the rate of increase of the logarithm of envelope. Therefore $N(E) = \lim_{x\to\infty}\frac{\theta(x)}{x\pi}$ and $\gamma(E) = \lim_{x\to\infty}\frac{\xi(x)}{x}$, where we have omitted average thanks to self-averaging. These expressions give the most simple

way to obtain spectrum and localization length from a practical point of view (for numerical calculations).

Phase and envelope obey the differential equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = k - \frac{V(x)}{k}\sin^2\theta + \phi(x)\sin 2\theta \qquad (19)$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{V(x)}{2k}\sin 2\theta - \phi(x)\cos 2\theta. \qquad (20)$$

Negative part of the spectrum: $E = -k^2$. If we perform the same manipulations with $E = -k^2$, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = k\cos 2\theta - \frac{V(x)}{k}\sin^2\theta + \phi(x)\sin 2\theta \qquad (21)$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = k\sin 2\theta + \frac{V(x)}{2k}\sin 2\theta - \phi(x)\cos 2\theta. \qquad (22)$$

Invariant measure for the phase. Using (B.1) we can write (for a positive energy)

$$\mathrm{d}\theta \stackrel{\text{(law)}}{=} k\,\mathrm{d}x + \tilde{\beta}(\theta)\,\mathrm{d}W(x) \qquad \text{(Stratonovich)}, \qquad (23)$$

where $\tilde{\beta}(\theta) = \sqrt{\frac{\sigma}{k^2}\sin^4\theta + g\sin^2 2\theta}$. The related FPE reads $\partial_x P(\theta; x) = F_\theta P(\theta; x) =$ where $F_\theta = -k\partial_\theta + \frac{1}{2}[\partial_\theta\tilde{\beta}(\theta)]^2$ is the forward generator. The current of the phase through the interval $[0, \pi]$ is the number of zeros of the wavefunction per unit length $N(E)$. The stationary solution for constant current $N(E) = \left[k - \frac{1}{2}\tilde{\beta}(\theta)\partial_\theta\tilde{\beta}(\theta)\right]P(\theta)$ is

$$P(\theta) = \frac{2N(E)}{\tilde{\beta}(\theta)}\int_\theta^\pi \frac{\mathrm{d}\theta'}{\tilde{\beta}(\theta')}\,\mathrm{e}^{2k\int_{\theta'}^\theta \frac{\mathrm{d}\theta''}{\tilde{\beta}(\theta'')^2}}, \qquad (24)$$

the IDoS is given by normalizing the distribution.

2.3. From multiplicative to additive noise

We have obtained the expression of the IDoS, which is given by normalizing the distribution (14) or the distribution (24) and is expressed as a double integral. The analysis of the random process and of its distribution is however made more simple by converting the SDE for the Ricatti variable (11) or the phase (23), which include multiplicative noises, into a SDE with additive noise. For that purpose we perform the following change of variable:

$$z \stackrel{\text{def}}{=} \sqrt{\frac{\sigma}{4g}}\sinh\varphi \qquad (25)$$

(that maps $\mathbb{R}$ to $\mathbb{R}$). The relation with the phase variable is $\cotg\theta = \sqrt{\frac{\sigma}{4g|E|}}\sinh\varphi$. The new variable obeys the SDE,

$$\mathrm{d}\varphi = -\sqrt{\frac{\sigma}{4g}}\left[\cosh\varphi + \left(\frac{4gE}{\sigma} - 1\right)\frac{1}{\cosh\varphi}\right]\mathrm{d}x + \sqrt{4g}\,\mathrm{d}W(x) = -U'(\varphi)\,\mathrm{d}x + \sqrt{4g}\,\mathrm{d}W(x), \qquad (26)$$

where we introduced the potential

$$\boxed{U(\varphi) = \sqrt{\frac{\sigma}{4g}}\left[\sinh\varphi + \left(\frac{4gE}{\sigma} - 1\right)\arctan(\sinh\varphi)\right]}. \qquad (27)$$

Note that $U(\varphi) = \mathcal{U}\left(z = \sqrt{\frac{\sigma}{4g}}\sinh\varphi\right)$.

In order to get the IDoS we construct the stationary solution of the FPE $\partial_x \mathcal{P}(\varphi; x) = F_\varphi \mathcal{P}(\varphi; x)$ where $F_\varphi = \partial_\varphi U'(\varphi) + 2g \partial_\varphi^2$ is the forward generator. The stationary solution for a constant current $-N(E)$ (the variable φ goes from $+\infty$ to $-\infty$, therefore currents for the phase θ and for φ are opposite) reads

$$\mathcal{P}(\varphi) = \frac{N(E)}{2g} \, \mathrm{e}^{-\frac{1}{2g} U(\varphi)} \int_{-\infty}^{\varphi} \mathrm{d}\varphi' \, \mathrm{e}^{\frac{1}{2g} U(\varphi')}, \tag{28}$$

which can also be directly obtained from (14) or (24) since $\mathrm{d}\varphi = -\sqrt{4g} \frac{\mathrm{d}\theta}{\beta(\theta)} = \sqrt{4g} \frac{\mathrm{d}z}{\beta(z)}$.

An alternative way to obtain the IDoS, which will help the discussion and will be used later, is to introduce the nth moment of the 'time' x needed by the process $\varphi(x)$ to reach $-\infty$, starting from $\varphi(0) = \varphi$ (spatial coordinate x plays the role of the 'time' and variable φ of the position). This problem is a first exit problem [38]. The moments are given by solving the equation $B_\varphi T_n(\varphi) = -n T_{n-1}(\varphi)$ where $B_\varphi = -U'(\varphi)\partial_\varphi + 2g \partial_\varphi^2$ is the backward Fokker–Planck generator. The solution is constructed for absorbing boundary condition at $-\infty$ and reflecting boundary at $+\infty$: $T_n(-\infty) = 0$ and $\partial_\varphi T_n(+\infty) = 0$. We find (see [38] or the appendix of [33])

$$T_n(\varphi) = \frac{n}{2g} \int_{-\infty}^{\varphi} \mathrm{d}\varphi' \, \mathrm{e}^{\frac{1}{2g} U(\varphi')} \int_{\varphi'}^{+\infty} \mathrm{d}\varphi'' \, \mathrm{e}^{-\frac{1}{2g} U(\varphi'')} T_{n-1}(\varphi''). \tag{29}$$

$T_n(+\infty)$ corresponds to the nth moment of the time needed by random process φ to cross $\mathbb{R}$; therefore, the moment of the distance ℓ between two consecutive nodes of the wavefunction $\psi(x; E)$. Let us emphasize on this point. We call ℓ_i the distance between the two consecutive nodes of the wavefunction: $\psi(0) = \psi(\ell_1) = \psi(\ell_1 + \ell_2) = \cdots = 0$. The problem of first exist problem is defined as $\varphi(x_i) = +\infty$ and $\varphi(x_i + \ell_i) = -\infty$ with $\varphi(x)$ finite for $x \in \,]x_i, x_i + \ell_i[$. The random variable is ℓ_i and $\langle \ell^n \rangle \equiv T_n(+\infty)$. Note that all distances are i.i.d. due to the fact that potential have a vanishing correlation length[8].

The IDoS per unit length is the average number of nodes of $\psi(x; E)$ per unit length, what corresponds to the inverse average distance between two consecutive nodes,

$$N(E)^{-1} = T_1(+\infty); \tag{30}$$

therefore

$$\boxed{N(E)^{-1} = \frac{1}{2g} \int_{-\infty}^{+\infty} \mathrm{d}\varphi \, \mathrm{e}^{\frac{1}{2g} U(\varphi)} \int_{\varphi}^{+\infty} \mathrm{d}\varphi' \, \mathrm{e}^{-\frac{1}{2g} U(\varphi')}} \tag{31}$$

that coincides with the normalization of the distribution (28). We will extract limiting behaviors of this exact expression by analyzing more precisely the dynamics of the random process $\varphi(x)$.

We first remark that the derivative of the potential at the origin is

$$U'(0) = \sqrt{\frac{4g}{\sigma}} E. \tag{32}$$

For $E > 0$ the potential is monotonous.

For $E < 0$ it develops a local minimum able to trap the process during a finite 'time'. In this latter case the local minimum of the potential is at $\varphi_+ > 0$ and the top of the barrier at $\varphi_- = -\varphi_+$,

$$\sinh \varphi_\pm = \pm \sqrt{\frac{4g|E|}{\sigma}}. \tag{33}$$

---

[8] In general, distances ℓ_i are decorrelated if correlation length is smaller than the length over which deterministic dynamics drives $\varphi(x)$ to ∞.

We easily check that $U''(\varphi_\pm) = \pm 2\sqrt{-E}$ for $E < 0$.

Important energy scales. We will identify later the relevant energy scales in the problem. In each regime ($g^3 \ll \sigma$ or $g^3 \gg \sigma$) two energy scales matter: the two largest scales among σ/g, $\sigma^{2/3}$, $\sqrt{g\sigma}$ and g^2.

- For small supersymmetric noise $g^3 \ll \sigma$, the two relevant energy scales are $\sigma^{2/3}$ and σ/g.
- For large supersymmetric noise $g^3 \gg \sigma$, the two energy scales are $\sqrt{g\sigma}$ and g^2.

2.4. Density of states for positive energies for $g^3 \gg \sigma$

The supersymmetric Hamiltonian is characterized by a purely positive spectrum (which follows from the structure $H_{\text{susy}} = Q^\dagger Q$) which presents the famous Dyson singularity at zero energy [15, 25],

$$N^{(\sigma=0)}(E) \sim \frac{g}{\ln^2(g^2/E)} \qquad \text{for} \quad E \to 0, \tag{34}$$

therefore it vanishes at zero energy: $N^{(\sigma=0)}(E = 0) = 0$. What is the fraction of states that migrate to $\mathbb{R}^-$ when a very small white noise $V(x)$ breaking the supersymmetry is added to H_{susy}?

Band center: $|E| \ll \sqrt{g\sigma}$. In the SDE (26), the exponential nature of the potential allows a decoupling of the deterministic force and the Langevin force (this works for $g^3 \gg \sigma$ only). We introduce the value for which the two forces are of the same order: $|U'(\Phi_0)| \overset{\text{def}}{=} 4g$,

$$\Phi_0 \simeq \ln(16\sqrt{g^3/\sigma}). \tag{35}$$

In the interval $[\Phi_0, +\infty[$, the dynamics of the random process is governed by the deterministic force. The process, starting from $+\infty$, reaches Φ_0 very fast. Then its dynamic is governed by the Langevin force in $[-\Phi_0, \Phi_0]$. Upon arrival at $-\Phi_0$ it is driven very fast to $-\infty$ by the deterministic force. This allows us to map the problem to the problem of free diffusion $d\varphi \simeq \sqrt{4g}\, dW(x)$ on the interval $[-\Phi_0, \Phi_0]$, with the reflecting boundary condition at $+\Phi_0$ and absorbing boundary condition at $-\Phi_0$ (a similar approximation was used in [33] to study the supersymmetric Hamiltonian at finite energy $E \ll g^2$). We immediately conclude that the average 'time' is $N(0)^{-1} = T_1(+\infty) \simeq \frac{(\text{distance})^2}{\text{diffusion}} = \frac{1}{4g}(2\Phi_0)^2$. Therefore a fraction of states

$$N(0) \sim \frac{g}{\ln^2(g^3/\sigma)} \tag{36}$$

have migrated to $\mathbb{R}^-$.

Let us analyze the structure of the distribution $\mathcal{P}(\varphi)$, given by (28), in the low-energy limit. For $\varphi \lesssim -\Phi_0$, we have $|U'(\varphi)| \gg 2g$ therefore $e^{U(\varphi)/2g}$ is extremely small and the integral over φ' is dominated by the close neighborhood of φ,

$$\mathcal{P}(\varphi) \simeq \frac{N(E)}{|U'(\varphi)|} \simeq \frac{N(E)}{4g}\frac{\cosh\Phi_0}{\cosh\varphi} \sim e^{\varphi+\Phi_0} \qquad \text{for} \quad \varphi \lesssim -\Phi_0. \tag{37}$$

This approximation reflects the fact that when deterministic evolution dominates $\text{Velocity}(\varphi) = \frac{d\varphi}{dx} \simeq -U'(\varphi)$ the distribution is $\mathcal{P}(\varphi) \propto 1/|\text{Velocity}(\varphi)|$.

In the intermediate interval $[-\Phi_0, \Phi_0]$, $e^{U(\varphi)/2g}$ is almost flat and the distribution is linear in this interval,

$$\mathcal{P}(\varphi) \simeq \frac{N(E)}{2g}\left[\frac{1}{2}e^{[U(-\Phi_0)-U(\varphi)]/2g} + (\varphi + \Phi_0)\right] \sim \varphi + \Phi_0 \qquad \text{for} \quad -\Phi_0 \lesssim \varphi \lesssim +\Phi_0,$$
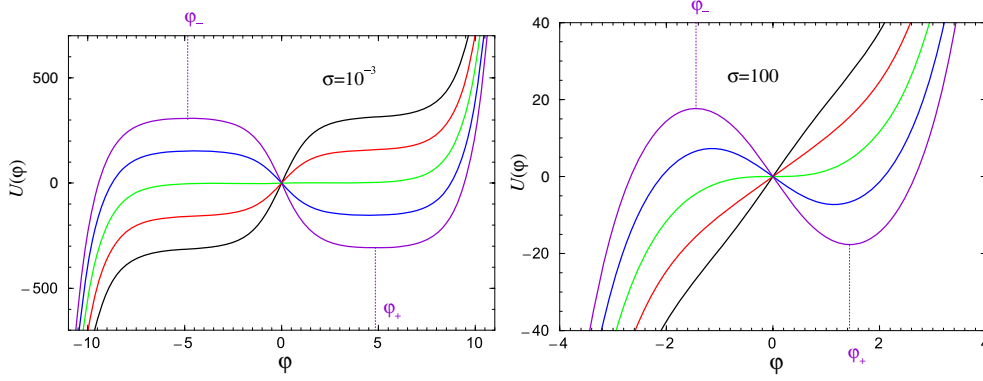
$$\tag{38}$$

Figure 1. Potential $U(\varphi)$ for $g = 1$. Left: $\sigma = 0.001$ with energies $E = 10, 5, 0, -5, -10$. Right: $\sigma = 100$ with energies $E = 100, 50, 0, -50, -100$.

where the first term is the contribution of the interval $]-\infty, -\Phi_0]$ to the integral (28). Finally, we find for the last interval

$$\mathcal{P}(\varphi) \simeq \frac{N(E)}{2g} \left[\frac{1}{2} e^{[U(-\Phi_0)-U(\varphi)]/2g} + (\varphi + \Phi_0) e^{[U(\Phi_0)-U(\varphi)]/2g} + \frac{1}{2} \frac{\cosh \Phi_0}{\cosh \varphi} \right] \quad \text{for} \quad \Phi_0 \lesssim \varphi. \tag{39}$$

It decreases exponentially: $\mathcal{P}(\varphi) \sim e^{-\varphi+\Phi_0}$. The curve is plotted on figure 2. Adding times spent in the three intervals gives the normalization $N(0)^{-1} \simeq \frac{1}{4g} + \frac{1}{g} \ln^2(16\sqrt{g^3/\sigma}) + \frac{1}{4g}$,

$$\boxed{N(E) \simeq \frac{4g}{\ln^2(2^8 g^3/\sigma) + 2}} \qquad \text{for} \quad |E| \ll \sqrt{g\sigma}. \tag{40}$$

Intermediate energies: $\sqrt{g\sigma} \ll E \ll g^2$. In this limit, the potential develops a double plateaux structure as suggested on figure 1. Once again we use the fact that the deterministic force depends exponentially on φ to decouple the effects of the Langevin force and the deterministic force. The equation $|U'(\varphi)| = 4g$ possesses now four solutions: $\varphi = \pm\Phi_0$ defined above and $\varphi = \pm\Phi_E$ with

$$\Phi_E \simeq \ln(E/\sqrt{\sigma g}). \tag{41}$$

The Langevin force dominates the evolution in intervals corresponding to plateaux of $U(\varphi)$, of width $\Phi_0 - \Phi_E \simeq \ln(16g^2/E)$, while the deterministic force governs the evolution on the other intervals. Let us follow the evolution of the process $\varphi(x)$. (i) In the interval $[\Phi_0, \infty[$ the deterministic force, $d\varphi \simeq -U'(\varphi)\, dx \simeq -\sqrt{\frac{\sigma}{4g}} \cosh \varphi\, dx$, drives the process from $\varphi = \infty$ to $\varphi = \Phi_0$ in a 'time' $1/(4g)$. (ii) In $[\Phi_E, \Phi_0]$, the Langevin force dominates: $d\varphi \simeq \sqrt{4g}\, dW(x)$. Given that one is reflected at Φ_0, the average 'time' required to reach Φ_E for the first time is $\frac{(\text{distance})^2}{\text{diffusion}} = \frac{1}{4g}(\Phi_0 - \Phi_E)^2$. (iii) In $[-\Phi_E, \Phi_E]$, the deterministic force dominates $d\varphi \simeq -U'(\varphi)dx \simeq -\sqrt{\frac{4g}{\sigma}} E \frac{1}{\cosh \varphi}\, dx$ and drives the process from one edge of the interval to the other in a 'time' $1/(2g)$. (iv) In $[-\Phi_0, -\Phi_E]$, the Langevin force dominates: the process crosses the interval in an average 'time' $\frac{1}{4g}(\Phi_0 - \Phi_E)^2$. (v) Finally, the deterministic force brings the process from $-\Phi_0$ to $-\infty$ in a 'time' $1/(4g)$.
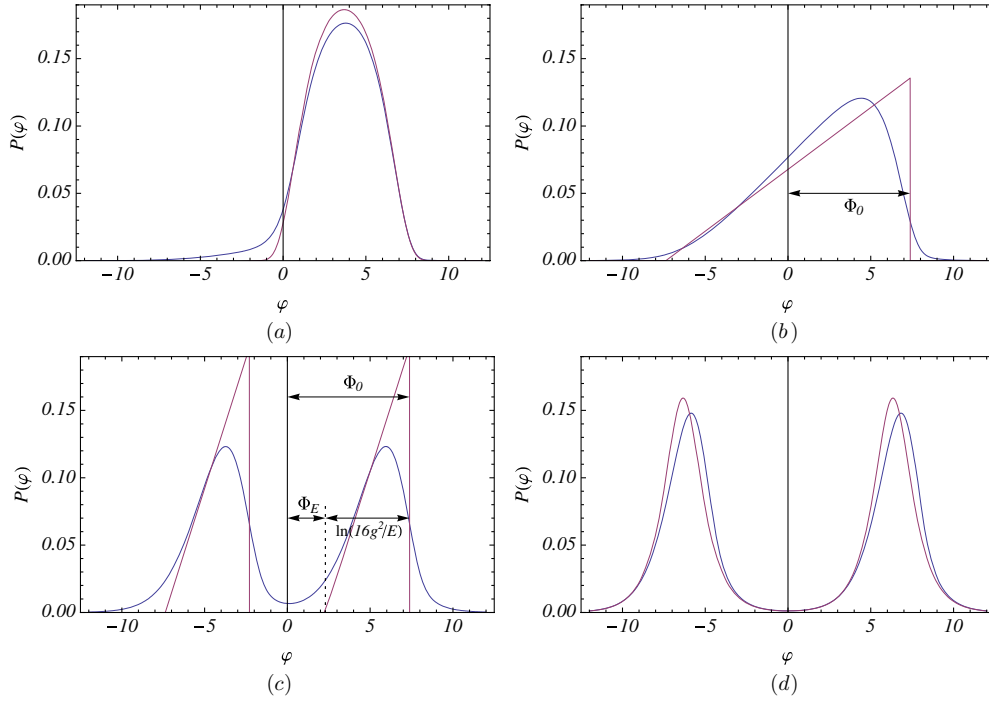
Figure 2. Distribution $\mathcal{P}(\varphi)$ (blue curves) for (a) $E = -0.01$, (b) $E = 0$, (c) $E = 0.1$ and (d) $E = 2$ (with $\sigma = 10^{-4}$ and $g = 1$). These behaviors can be understood from the shapes of the potential of the left part of figure 1. For $E = -\sqrt{\sigma g} = -0.01$: the distribution is compared to $\exp[-\frac{1}{g}\sqrt{|E|}\cosh(\varphi - \varphi_+)]$. For the other energies, the distribution is compared to approximations obtained in the text.

The analysis of the distribution (28) follows the same logic. In the two intervals where motion is diffusive (where the process spends most of the time),

$$\mathcal{P}(\varphi) \simeq \frac{N(E)}{2g} \begin{cases} \varphi + \Phi_0 & \text{for} \quad -\Phi_0 \lesssim \varphi \lesssim -\Phi_E \\ \varphi - \Phi_E & \text{for} \quad \Phi_E \lesssim \varphi \lesssim \Phi_0 \end{cases} \tag{42}$$

(see figure 2). Normalizing this distribution gives $N(E)^{-1} \simeq \frac{1}{g} + 2\frac{(\Phi_0 - \Phi_E)^2}{4g} \simeq \frac{1}{2g}\ln^2(16g^2/E)$; therefore, we recover the usual Dyson singularity (the scalar potential $V(x)$ plays no role),

$$\boxed{N(E) \simeq \frac{2g}{\ln^2(16g^2/E) + 2}} \qquad \text{for} \quad \sqrt{g\sigma} \ll E \ll g^2. \tag{43}$$

Large energies: $E \gg g^2$. Finally, for completeness, we give the distribution in the high-energy limit. In this case, the phase distribution is almost flat $P(\theta) \simeq 1/\pi$ therefore the distribution for φ presents the double peak structure,

$$\mathcal{P}(\varphi) \simeq \frac{1}{\pi} \frac{\sinh \varphi_+ \cosh \varphi}{\sinh^2 \varphi_+ + \sinh^2 \varphi}, \tag{44}$$

where φ_+ is defined by (33). The two peaks are associated with inflection points of the potential $U(\varphi)$ where the force is minimum (note that $\pm\tilde{\varphi}_+ \simeq \varphi_\pm$). The IDoS is given by the free IDoS $N(E) \simeq \frac{1}{\pi}\sqrt{E}$.

2.5. Lifshits tail

In this paragraph we analyze the tail of the IDoS in the region of rarefaction of states, that is for $E \to -\infty$.

For negative energies, the process $\varphi(x)$ is trapped by the well at $\varphi = \varphi_+$ a very long 'time' where positions $\varphi_\pm$ of the extrema of the potential are given by (33). The average 'time' needed to exit the well due to a fluctuation (Langevin force) is given by the Arrhenius formula. The height of the potential barrier is given by

$$\frac{1}{2g}[U(\varphi_-) - U(\varphi_+)] = \sqrt{\frac{\sigma}{4g^3}} F\left(\frac{4g|E|}{\sigma}\right) \tag{45}$$

with

$$
\begin{aligned}
F(x) &\stackrel{\text{def}}{=} 2(x+1)[\arctan(\sqrt{x+1} + \sqrt{x}) - \pi/4] - \sqrt{x} \\
&= \begin{cases} \dfrac{2}{3}x^{3/2} + O(x^{5/2}), & \text{for} \quad x \ll 1 \\ \dfrac{\pi}{2}x - 2\sqrt{x} + \dfrac{\pi}{2} + O(x^{-1/2}), & \text{for} \quad x \gg 1. \end{cases}
\end{aligned} \tag{46}
$$

Assuming $\frac{1}{2g}[U(\varphi_-) - U(\varphi_+)] \gg 1$ we can expand integrands in (31). As we can see on figure 1, in the limit $g^3 \ll \sigma$ the potential $U(\varphi)$ is parabolic near its extrema and we can use formula (A22) of [33]. However in the regime $g^3 \gg \sigma$ the parabolic approximation is not correct. In this latter case, noting that

$$U(\varphi) \underset{\varphi \sim \varphi_\pm}{\simeq} \text{const.} \pm 2\sqrt{|E|}\cosh(\varphi - \varphi_\pm), \tag{47}$$

we obtain

$$\boxed{N(E) \simeq \frac{g}{2}\left[e^{\frac{\sqrt{|E|}}{g}} K_0\left(\frac{\sqrt{|E|}}{g}\right)\right]^{-2} \exp-\sqrt{\frac{\sigma}{4g^3}} F\left(\frac{4g|E|}{\sigma}\right)}$$

$$\text{for} \qquad |E| \gg \max(\sigma^{2/3}, \sqrt{g\sigma}), \tag{48}$$

where $K_0(z)$ is the MacDonald function (modified Bessel function of third kind). We can now consider two situations, depending on which among the supersymmetric noise $\phi(x)$ or the scalar noise $V(x)$ dominates.

Small supersymmetric noise: $g^3 \ll \sigma$. In the intermediate range we recover from (48) the Lifshits tail of the Hamiltonian $H_{\text{scalar}} = -\frac{d^2}{dx^2} + V(x)$ [5, 12, 39],

$$N(E) \simeq \frac{\sqrt{|E|}}{\pi} \exp-\frac{8|E|^{3/2}}{3\sigma} \qquad \text{for} \quad \sigma^{2/3} \ll |E| \ll \sigma/g. \tag{49}$$

The supersymmetric noise does not affect the DoS in this regime.

For larger values of $|E|$ the tail takes the form

$$N(E) \simeq \frac{\sqrt{|E|}}{\pi} \exp\left[-\frac{\pi|E|}{\sqrt{g\sigma}} + 2\frac{\sqrt{|E|}}{g} - \frac{\pi}{4}\sqrt{\frac{\sigma}{g^3}}\right] \qquad \text{for} \quad |E| \gg \sigma/g. \tag{50}$$

Even though the supersymmetric noise is much smaller than $V(x)$, the behavior at largest values of $|E|$ is due to a competition between ϕ and V.

Large supersymmetric noise: $g^3 \gg \sigma$. Expanding (48), we see that the IDoS presents the limiting behaviors

$$N(E) \simeq \frac{2g}{\ln^2(g^2/|E|)} \exp-\frac{\pi|E|}{\sqrt{g\sigma}} \qquad \text{for} \quad \sqrt{g\sigma} \ll |E| \ll g^2 \tag{51}$$

and

$$N(E) \simeq \frac{\sqrt{|E|}}{\pi} \exp\left[-\frac{\pi|E|}{\sqrt{g\sigma}} + 2\frac{\sqrt{|E|}}{g} \right] \qquad \text{for} \quad |E| \gg g^2. \tag{52}$$

It is interesting to note that the prefactors coincide with the limiting behaviors obtained for positive energies: $N(E) \simeq \frac{2g}{\ln^2(g^2/E)}$ for $\sqrt{g\sigma} \ll E \ll g^2$ and $N(E) \simeq \frac{1}{\pi}\sqrt{E}$ for $E \gg g^2$.

2.6. Extreme value spectral statistics

Up to now we have studied spectral properties through the density of states. In this section, we consider another property of the spectrum: the problem of extreme value statistics for the eigenvalues of the Hamiltonian (2). Let us formulate the problem: for a given realization of the potential, the spectral (Sturm–Liouville) problem $H\psi(x) = E\psi(x)$ for boundary conditions $\psi(0) = \psi(L) = 0$ has a discrete set of solutions $\mathrm{Spec}(H) = \{E_n\}$ (we assume that label corresponds to rank the eigenvalues as $E_1 < E_2 < E_3 < \cdots$). We ask the question: what is the distribution

$$W_n(E) = \langle \delta(E - E_n) \rangle \tag{53}$$

of the nth eigenvalue? These distributions give a much more precise information on the spectrum than the density of states, what is already clear from the relation $\sum_{n=1}^{\infty} W_n(E) = L\rho_L(E)$ where $\rho_L(E)$ is the average DoS per unit length accounting for the Dirichlet boundary conditions at $x = 0$ and $x = L$ (when $L \to \infty$ the sensitivity to the boundary conditions disappears: $\lim_{L\to\infty} \rho_L(E) = N'(E)$, where $N(E)$ is the IDoS per unit length of the infinite system studied above). The distribution $W_n(E)$ gives the probability to find the nth eigenvalue at E whereas the DoS $\rho_L(E)$ tells us the probability to find *any* eigenvalue at E.

The study of extreme value statistics in various contexts has attracted a lot of attention. Extreme value statistics of uncorrelated and identically distributed variables were classified long-time ago (Gumbel for an exponentially decreasing distribution, Fréchet for a power law and Weibull for distribution with bounded support [40, 41]). Extreme value statistics for correlated variables is a much more difficult task. A famous example is the Tracy–Widom distribution for eigenvalues of Gaussian random matrices [42, 43]. There has been a renewed interest in such problems in the last few years (see, for example, [44, 45]).

The question of extreme value statistics of a 1D random Hamiltonian was first addressed in [46] for the Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_n v_n \delta(x - x_n)$ where positions are uncorrelated and uniformly distributed; weights v_n are positive, uncorrelated and distributed according to a Poisson law. The case of the Hamiltonian $H_{\mathrm{scalar}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ where $V(x)$ is a white noise was studied in [47] where $W_1(E)$ was derived. This result was generalized in [33] where it was shown that the distributions $W_n(E)$ are Gumbel laws when $L \to \infty$: despite that eigenvalues E_n are random variables *a priori* correlated, extreme value distributions coincide with those of *uncorrelated* variables. Such an absence of spectral correlations is a consequence of the strong localization of the wavefunctions in this regime [48]. Extreme value spectral statistics for the supersymmetric Hamiltonian (1) near the delocalization transition was also considered in [33]; it was noted that in this case the distributions $W_n(E)$ do not coincide with extreme value statistics for uncorrelated variables, a consequence of spectral correlations near the delocalization transition.

We first consider the limit of strong supersymmetric disorder $g^3 \gg \sigma$. When a small scalar noise is added to the supersymmetric Hamiltonian, we have seen that the spectrum is not anymore constrained to be in $\mathbb{R}^+$. A simple way to obtain the typical ground-state energy is to write that $LN(E_1) \sim 1$ from which we obtain $E_1 \sim -\sqrt{\sigma g} \ln L$. Since the exponential

tail of the IDoS is usually associated with strongly localized states, what will be supported by the study of localization in section 3, we expect that the distributions (53) are similar to the one obtained for H_{scalar} (Gumbel laws). This is the aim of the following paragraph to show this statement explicitly.

We first assume that the length of the system is sufficiently long so that the support of (53) is in $\mathbb{R}^-$ with energies far from the band center $|E| \gg \sqrt{g\sigma}$. In this case, the process $\varphi(x)$ is trapped by the well of the potential $U(\varphi)$. The 'time' ℓ needed by the process to go from $+\infty$ to $-\infty$ (ℓ is the distance between two consecutive nodes of the wavefunction) is dominated by the time needed to exit the well. Its moments (29) are given by $\langle \ell^n \rangle = T_n(+\infty) \simeq n![T_1(+\infty)]^n$ what corresponds to a Poisson law. As a consequence it was shown in [33] that

$$W_n(E) \simeq L\rho(E)\frac{[LN(E)]^{n-1}}{(n-1)!}\,\mathrm{e}^{-LN(E)},\tag{54}$$

where the IDoS per unit length of the infinite system is given by (51), (52). The question of which, among (51) or (52), is the behavior to be considered in order to analyze $W_n(E)$ depends on where the support of the distribution is. *A priori* for the longest size $L \to \infty$ we expect that $W_n(E)$ has its support for energies below $-g^2$ whereas for intermediate length L the support is between $-g^2$ and $-\sqrt{\sigma g}$. This question will be rediscussed more precisely below.

We see from equations (51), (52) that the density of states per unit length is well approximated by $\rho(E) \simeq \frac{\pi}{\sqrt{g\sigma}}N(E)$. In a first time we assume that L is sufficiently large so that the support of $W_n(E)$ is below $-g^2$. We can use (52) from which we write

$$W_n(E) \simeq \frac{1}{(n-1)!}\frac{\pi}{\sqrt{g\sigma}}\left(\frac{L}{\pi}\right)^n \mathrm{e}^{-f(E)}\tag{55}$$

with

$$f(E) = -\frac{n}{2}\ln|E| + \frac{n\pi|E|}{\sqrt{g\sigma}} - \frac{2n\sqrt{|E|}}{g} + n\frac{\pi}{4}\sqrt{\frac{\sigma}{g^3}} + \frac{\sqrt{|E|}L}{\pi}\,\mathrm{e}^{-\frac{\pi|E|}{\sqrt{g\sigma}}+\frac{2\sqrt{|E|}}{g}-\frac{\pi}{4}\sqrt{\sigma/g^3}}\tag{56}$$

(note that we have reintroduced the term $\frac{\pi}{4}\sqrt{\sigma/g^3}$ neglected in (52) but present in (50); this will be useful to discuss the other limit $g^3 \ll \sigma$). It is convenient to re-scale energy and length as

$$y = \frac{\pi|E|}{\sqrt{g\sigma}} \qquad \text{and} \qquad \tilde{L} = \frac{(g\sigma)^{1/4}L}{\pi^{3/2}n},\tag{57}$$

and furthermore to introduce the quantity $\varepsilon = (\sigma/\pi^2 g^3)^{1/4}$. Hence, in terms of the new variables we find $f(E) = g(y) + \text{const}$ for

$$g(y) = n\left[-\tfrac{1}{2}\ln y + y - 2\varepsilon\sqrt{y} + \tilde{\mathcal{L}}\sqrt{y}\,\mathrm{e}^{-y+2\varepsilon\sqrt{y}}\right] \qquad \text{with} \quad \tilde{\mathcal{L}} = \tilde{L}\,\mathrm{e}^{-(\frac{\pi\varepsilon}{2})^2}.\tag{58}$$

The derivative reads $g'(y) = n\left(1 - \frac{1}{2y} - \frac{\varepsilon}{\sqrt{y}}\right)(1 - \tilde{\mathcal{L}}\sqrt{y}\,\mathrm{e}^{-y+2\varepsilon\sqrt{y}})$. The first parentheses vanish for a value of y corresponding to energy out of the range defined in (52); it should not be considered as an extremum. The extremum $y = \tilde{y}$ is solution of

$$\tilde{\mathcal{L}}\sqrt{\tilde{y}}\,\mathrm{e}^{-\tilde{y}+2\varepsilon\sqrt{\tilde{y}}} = 1.\tag{59}$$

In the limit $L \to \infty$ we find

$$\tilde{y} = \ln\tilde{L} + \frac{1}{2}\ln\ln\tilde{L} + 2\varepsilon\sqrt{\ln\tilde{L}} + \left(2 - \frac{\pi^2}{4}\right)\varepsilon^2 + O\left(\frac{\ln\ln\tilde{L}}{\sqrt{\ln\tilde{L}}}, \frac{\varepsilon}{\sqrt{\ln\tilde{L}}}\right).\tag{60}$$

We can easily show that higher derivatives are given by $g^{(k)}(\tilde{y}) \simeq n(-1)^k$ for $\tilde{L} \to \infty$ and $k > 1$. Typical value of the energies (value that maximizes $W_n(E)$) is

$$\boxed{E_n^{\text{typ}} = -\frac{\sqrt{g\sigma}}{\pi}\ln(\tilde{L}\sqrt{\ln\tilde{L}}) - \frac{2\sigma^{3/4}}{\pi^{3/2}g^{1/4}}\sqrt{\ln\tilde{L}} + \cdots}.\tag{61}$$
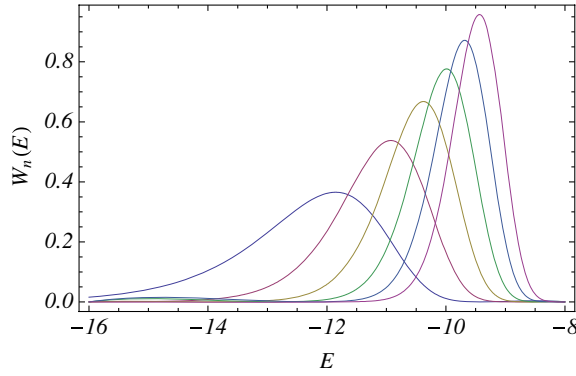
14

Figure 3. The nth energy-level distribution $W_n(E)$ for $1 \leqslant n \leqslant 6$ with $g = 10, \sigma = 1$ and $L = 10^4$. We see from equations (61), (62) that the dependence in n of the typical energy level and the width are $E_n^{\text{typ}} \simeq E_1^{\text{typ}} + \frac{1}{\pi}\sqrt{g\sigma}\ln(n)$ and $\delta E_n \simeq \delta E_1/\sqrt{n}$, respectively.

The width of the distribution is independent on the length,

$$\delta E_n \simeq \frac{1}{\pi}\sqrt{\frac{\sigma g}{n}}. \tag{62}$$

Following [33] we can reconstruct $g(y)$ in the neighborhood of $\tilde{y}$ by using the derivatives. An alternative formulation is to expand $g\left(\tilde{y} - \frac{1}{\sqrt{n}}X\right)$, where $X = \frac{1}{\delta E_n}\left(E - E_n^{\text{typ}}\right) = \sqrt{n}(\tilde{y} - y)$. We can neglect all terms vanishing in the limit $\tilde{y} \to \infty$ (i.e. $\tilde{L} \to \infty$) since $X \sim 1$. We have

$$\frac{1}{n}g\left(\tilde{y} - \frac{1}{\sqrt{n}}X\right) = -\frac{1}{2}\ln(\tilde{y} - X/\sqrt{n}) + \tilde{y} - X/\sqrt{n} - 2\varepsilon\sqrt{\tilde{y} - X/\sqrt{n}}$$
$$+ \tilde{L}\sqrt{\tilde{y} - X/\sqrt{n}}\,e^{-\tilde{y} + X/\sqrt{n} + 2\varepsilon\sqrt{\tilde{y} - X/\sqrt{n}}}. \tag{63}$$

Using (59), we obtain

$$\frac{1}{n}g\left(\tilde{y} - \frac{1}{\sqrt{n}}X\right) \underset{\tilde{y}\to\infty}{\simeq} \text{const} - \frac{X}{\sqrt{n}} + e^{X/\sqrt{n}}. \tag{64}$$

Therefore we have recovered the Gumbel law

$$W_n(E) \underset{L\to\infty}{=} \frac{1}{\delta E_n}\,\omega_n\left(\frac{E - E_n^{\text{typ}}}{\delta E_n}\right) \qquad \text{with} \qquad \omega_n(X) = \frac{n^{n-1/2}}{(n-1)!}\exp(\sqrt{n}X - n\,e^{X/\sqrt{n}}). \tag{65}$$

The first distributions are plotted on figure 3.

Let us do several remarks

- The results (61), (62), (65) have been obtained using the asymptotic form of the IDoS (52). Therefore, it was assumed from the outset of the calculation that the support of the distribution (65) is below $-g^2$. The condition $-E_n^{\text{typ}} \gg g^2$ can be recast as a condition on the length of the system

$$L \gg \frac{n\pi^{3/2}}{(\sigma g)^{1/4}}\,e^{\pi\sqrt{g^3/\sigma}}. \tag{66}$$
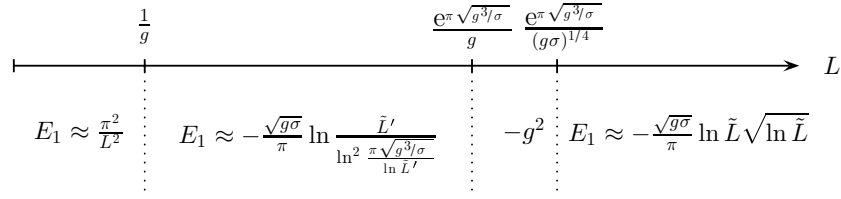
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$$\frac{1}{g} \qquad\qquad\qquad\qquad \frac{e^{\pi\sqrt{g^3/\sigma}}}{g} \quad \frac{e^{\pi\sqrt{g^3/\sigma}}}{(g\sigma)^{1/4}}$$

$$E_1 \approx \frac{\pi^2}{L^2} \qquad E_1 \approx -\frac{\sqrt{g\sigma}}{\pi} \ln \frac{\tilde{L}'}{\ln^2 \frac{\pi\sqrt{g^3/\sigma}}{\ln \tilde{L}'}} \qquad -g^2 \qquad E_1 \approx -\frac{\sqrt{g\sigma}}{\pi} \ln \tilde{L} \sqrt{\ln \tilde{L}}$$

Figure 4. Illustration of regimes for the typical ground-state energy E_1 with increasing system sizes L in the large supersymmetric noise limit $g^3 \gg \sigma$.

- If the length of the system does not satisfy condition (66), the support of $W_n(E)$ is shifted above to the interval between $-g^2$ and $-\sqrt{\sigma g}$. Therefore the above calculation should be redone starting from (51). The results are almost similar: the final distribution (65) still holds for the same width (62). Only the behavior of the typical energy changes slightly,

$$E_n^{\text{typ}} \simeq -\frac{\sqrt{g\sigma}}{\pi} \ln \left(\frac{\tilde{L}'}{\ln^2 \frac{\pi\sqrt{g^3/\sigma}}{\ln \tilde{L}'}} \right) \tag{67}$$

with $\tilde{L}' = \frac{2gL}{n}$. This expression holds when the length of the system is such that

$$\frac{1}{g} \ll L \ll \frac{1}{g} e^{\pi\sqrt{g^3/\sigma}}. \tag{68}$$

The crossover (for $L \sim e^{\pi\sqrt{g^3/\sigma}}$) obviously corresponds to $E_n^{\text{typ}} \sim -g^2$.

For smaller system sizes $L \lesssim 1/g$ we expect the disorder to have a perturbative effect and consequently the energy level to be close to the free levels $E_n \simeq (\pi n/L)^2$, $n \in \mathbb{N}^*$. Figure 4 summarizes the different regimes for the ground-state energy E_1.

- The introduction of the scalar noise has rather strong consequences on the distributions $W_n(E)$. (A) For $\sigma = 0$ distributions $W_n(E)$ are broad distributions (in particular $E_1^{\text{typ}} \sim g^2 e^{-gL}$ and $\langle E_1 \rangle \sim g^2 e^{-(gL)^{1/3}}$) departing from Gumbel distributions, a consequence of spectral correlations [33]. (B) For $\sigma \neq 0$ the distributions are narrow distributions centered on $E_n^{\text{typ}} \sim -\sqrt{g\sigma} \ln L$ and coinciding with Gumbel distributions, an indication of absence of spectral correlations. We now characterize the crossover scale of scalar noise separating the two situations (A) and (B). Let us reason at *fixed* g and L (with $L \gg 1/g$) and introduce an infinitesimal σ: we start from the situation (A). If σ is increased, the length L fulfills condition (68) and the ground-state energy is given by (67), provided that at least one state is below $-\sqrt{g\sigma}$. This last condition reads $LN(-\sqrt{g\sigma}) \sim \frac{gL}{\ln^2(g^3/\sigma)} \gtrsim 1$. This leads to the crossover value

$$\sigma_c \sim g^3 e^{-\sqrt{gL}} \tag{69}$$

separating (A) and (B). Below this value ($\sigma \lesssim \sigma_c$) the scalar noise can be ignored. Another simple way to obtain this scale is to write $-E_n^{\text{typ}} \gtrsim \sqrt{g\sigma}$, where the typical energy is given by (67).

- *Small supersymmetric noise.* Finally we mention the results for $\sigma \gg g^3$. If the support of $W_n(E)$ is in the interval between $-\sigma/g$ and $-\sigma^{2/3}$ the supersymmetric noise does not
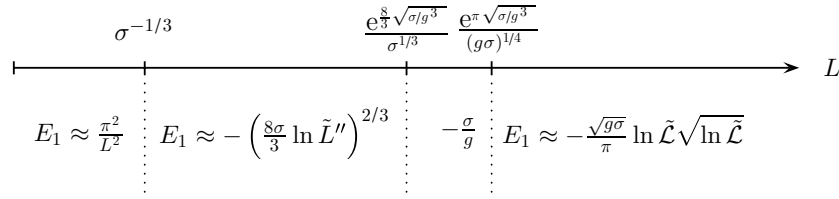
Figure 5. Illustration of regimes for the typical ground-state energy E_1 with increasing system sizes L in the small supersymmetric noise limit $g^3 \ll \sigma$. We recall that $\tilde{\mathcal{L}} = \tilde{L} \, \mathrm{e}^{-\frac{\pi}{4}\sqrt{\sigma/g^3}}$.

play any role and we recover the results of [33] obtained for H_{scalar}: the form (65) holds for

$$E_n^{\text{typ}} \simeq -\left(\frac{3\sigma}{8} \ln \tilde{L}''\right)^{2/3} \qquad \text{and} \qquad \delta E_n \simeq \frac{\sigma^{2/3}}{2\sqrt{n}} (\ln \tilde{L}'')^{-1/3} \qquad (70)$$

with $\tilde{L}'' = \frac{L\sigma^{1/3}}{2\pi n}$ (this time the width of the distributions vanish in the limit $L \to \infty$). The hypothesis made on the position of the support of $W_n(E)$ implies that the length satisfies

$$\sigma^{-1/3} \ll L \ll n\sigma^{-1/3} \, \mathrm{e}^{\frac{8}{3}\sqrt{\frac{\sigma}{g^3}}}. \qquad (71)$$

For longer lengths we recover the behaviors (61), (62). Now the term ε^2 of (60) neglected above is large, what adds a term to (61). We can check that at the crossover ($L \sim \mathrm{e}^{\frac{8}{3}\sqrt{\sigma/g^3}}$) both (61) and (70) give $E_n^{\text{typ}} \sim -\sigma/g$.

An illustration of the different regimes is given in figure 5.

3. Localization

Up to now we have concentrated ourselves on the spectral properties of the random Hamiltonian; however, the most stricking property of Hamiltonians with random potentials is the localization of their wavefunctions. In a typical situation, for example if we consider the Hamiltonian $H_{\text{scalar}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ where $V(x)$ is random with short-range correlations, one should distinguish two regions in the spectrum: in the low-energy regime, lowest energy states are those trapped by deep wells of the potential. The nature of the trapping depends on the statistical properties of $V(x)$ (Gaussian white noise, low density of repulsive or attractive impurities, etc). This kind of localization is rather natural. It is correlative to a rarefaction of states reflected in the Lifshits exponential tail of the IDoS (section 2.5). On the other hand, in the high-energy range ($E \gg$ disorder) the phenomenon of Anderson localization [49] takes place: in a regime where the static potential is *a priori* perturbative, due to interferences between the extremely large number of scattering paths, the wavefunctions decrease exponentially over distances larger than the Fermi wavelength, a nonperturbative effect. Whereas in the 3D situation a delocalization (Anderson) transition occurs by tuning the strength of the disordered potential [6], the 1D case is particular since all states are localized [50], a statement rigorously proved in [51, 52]. The problem of 1D Anderson localization has been reformulated and re-examined in many works (see, for example, [5, 37, 53, 61]). As we mentioned in the introduction the random supersymmetric Hamiltonian presents particular localization properties since the low-energy Dyson singularity of the IDoS [1] is accompanied by a delocalization transition [15, 20]. These features are strongly related to

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the (super)symmetry of the Hamiltonian. In this section, we will examine how the localization picture is modified by breaking the supersymmetry in the Hamiltonian (2).

Information on localization of wavefunctions can be obtained by considering different variables. The most transparent formulation is probably provided by considering the variables (θ, ξ) of the phase formalism. Localization length ℓ_{loc} is related to the damping rate of the envelope of the wavefunction. Therefore, we can define the localization length by analyzing the solution of the Cauchy problem: from equations (17), (18) we take as a definition the relation $1/\ell_{\mathrm{loc}} = \gamma = \lim_{x \to \infty} \frac{\xi(x)}{x}$, where γ is the Lypaunov exponent (note that we can omit the disorder averaging in this definition thanks to self-averaging of this process)[9]. It is interesting to emphasize that this definition of the localization length is extracted from the solutions $\psi(x; E)$ of the Cauchy problem, and not from the real wavefunctions $\varphi_n(x)$ (solution of the Sturm–Liouville problem). In other terms the Lyapunov exponent gives a good estimate of the localization length of $\varphi_n(x)$ if the statistical properties of the envelope of the solution of the Schrödinger equation is not affected when imposing the second boundary condition. In the high-energy limit where processes $\theta(x)$ and $\xi(x)$ rapidly decorrelate [13] this is not a problem, however it is not obvious that this holds in any situation (in particular for the supersymmetric Hamiltonian H_{susy}, the Lyapunov exponent does not seem to give a fully satisfactory information as pointed out in the conclusion of [33]).

Since the analysis provided in the previous sections is based on the study of the dynamics of $z(x)$ or $\varphi(x) = \mathrm{argsinh}(\sqrt{4g/\sigma}\, z(x))$, we will extract the localization length from the statistical properties of these stochastic processes. We will derive several formulae and use the most adapted in the various regimes. Let us recall that the simplest expression of the Lyapunov exponent is given by the average of the Ricatti variable [5],

$$\gamma(E) = \left\langle \frac{\psi'(x; E)}{\psi(x; E)} \right\rangle = \langle z \rangle + \langle \phi \rangle \,. \tag{72}$$

As in the previous sections we consider here the case $\langle \phi \rangle = 0$. Together with the expression of the stationary distribution $T(z)$, this immediately gives the Lyapunov exponent. Note that since $T(z) \simeq N(E)/z^2$ for $|z| \to \infty$ (Rice formula), the expression must be understood as $\gamma = \int_{\mathbb{R}} \mathrm{d}z\, z \frac{T(z) - T(-z)}{2}$ in order to deal with a well-defined integral. We can also avoid this problem by deriving other formulae, which is what we do now.

Positive part of the spectrum: $E = +k^2$. We rewrite the two SDE (19), (20) for phase and envelope as

$$\mathrm{d}\theta = k\, \mathrm{d}x - \frac{\sqrt{\sigma}}{k} \sin^2\theta\, \mathrm{d}W_1(x) + \sqrt{g} \sin 2\theta\, \mathrm{d}W_2(x) \qquad \text{(Stratonovich)} \tag{73}$$

$$\mathrm{d}\xi = \frac{\sqrt{\sigma}}{2k} \sin 2\theta\, \mathrm{d}W_1(x) - \sqrt{g} \cos 2\theta\, \mathrm{d}W_2(x) \qquad \text{(Stratonovich)}, \tag{74}$$

where $W_1(x)$ and $W_2(x)$ are two normalized independent Wiener processes. Since the Lyapunov exponent is related to $\langle \xi(x) \rangle$ we connect these Stratonovich-SDE to some Ito-SDE and use the fact that with this latter prescription, random process and noise are decorrelated at equal 'time',

$$\mathrm{d}\theta = \left(k + \frac{\sigma}{2k^2} \sin^2\theta \sin 2\theta + \frac{g}{2} \sin 4\theta\right) \mathrm{d}x - \frac{\sqrt{\sigma}}{k} \sin^2\theta\, \mathrm{d}W_1 + \sqrt{g} \sin 2\theta\, \mathrm{d}W_2 \quad \text{(Ito)} \tag{75}$$

---

[9] This picture suggests that the wavefunction behaves roughly as $\psi(x) \sim \mathrm{e}^{\pm \gamma x} \times$ (oscillations); however, one should keep in mind that such a simple picture is dangerous since it forgets the important fact that the argument of the exponential, $\xi(x)$, presents large fluctuations increasing like $\sqrt{x}$ (fluctuations of $\xi(x)/x$ vanish for $x \to \infty$, but not those of $\xi(x)$). The envelope of the wavefunction is an exponential of a drifted Brownian motion, what can have important consequences [54]; neglecting this important feature can lead to a wrong conclusion, like in [55].

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$$\mathrm{d}\xi = \left(-\frac{\sigma}{2k^2}\sin^2\theta\cos 2\theta + g\sin^2 2\theta\right)\mathrm{d}x + \frac{\sqrt{\sigma}}{2k}\sin 2\theta\,\mathrm{d}W_1 - \sqrt{g}\cos 2\theta\,\mathrm{d}W_2 \quad (\text{Ito}). \quad (76)$$

We immediately obtain the following expression:

$$\gamma = \frac{\mathrm{d}\langle\xi\rangle}{\mathrm{d}x} = -\frac{\sigma}{2k^2}\langle\sin^2\theta\cos 2\theta\rangle + g\langle\sin^2 2\theta\rangle, \quad (77)$$

where averaging is realized with the stationary distribution. This relation is similar in spirit to the one derived in [56] for H_{scalar}. This equation, with the distribution (24), gives another explicit expression for the Lyapunov exponent. The Lyapunov exponent can also be expressed in term of the distribution (14),

$$\gamma = \frac{\sigma}{2}\left\langle\frac{E + \left(\frac{8Eg}{\sigma}-1\right)z^2}{(E+z^2)^2}\right\rangle \quad (78)$$

or the distribution (28)

$$\gamma = 2g\langle\Upsilon_{4Eg/\sigma}(\varphi)\rangle \qquad \text{with} \qquad \Upsilon_A(\varphi) \stackrel{\text{def}}{=} \frac{A + (2A-1)\sinh^2\varphi}{(A+\sinh^2\varphi)^2}. \quad (79)$$

Note that expressions (77)–(79) are valid for $E > 0$ and are note appropriate to study the limit $E \to 0$: for example the equation with the Ricatti variable would take the absurd form '$\gamma = -\frac{\sigma}{2}\langle\frac{1}{z^2}\rangle$' (absurd since $T(z)$ is regular at $z = 0$). The origin of the problem can be understood from (12) that shows that in the limit $E \to 0$, the two terms $T(z) = \frac{N(E)}{z^2+E} - \frac{\beta(z)}{2(z^2+E)}\frac{\mathrm{d}}{\mathrm{d}z}[\beta(z)T(z)]$ cannot be considered separately. A more detailed discussion is given in appendix C.

Band center. In this regime, due to the previous remark, we start from $\gamma = \langle z\rangle = \sqrt{\frac{\sigma}{4g}}\langle\sinh\varphi\rangle$. Using the fact that the approximate expression of the distribution is symmetric for $|\varphi| \gtrsim \Phi_0$, we write $\langle\sinh\varphi\rangle \simeq \int_{-\Phi_0}^{+\Phi_0}\mathrm{d}\varphi\,\mathcal{P}(\varphi)\sinh\varphi$. We obtain

$$\boxed{\gamma(0) \simeq \frac{4g}{\ln(16g^3/\sigma)}} \qquad \text{for} \quad |E| \ll \sqrt{g\sigma}. \quad (80)$$

Note however that the multiplicative factor 4 is directly related to our definition of Φ_0 separating regions where deterministic force and Langevin force dominates: $|U'(\Phi_0)| = 4g$. Therefore, in this derivation, the factor 4 is arbitrary. However the replica method of section 4 will predict the same prefactor. We would not have the same problem for the other regime since we will use formula (79) instead of (72).

This result shows that even a tiny $\sigma \to 0$ scalar noise is sufficient to lift the delocalization transition of the supersymmetric Hamiltonian.

Intermediate energies: $\sqrt{g\sigma} \ll E \ll g^2$. The function $\Upsilon_A(\varphi)$ presents two symmetric peaks centered on $\varphi \simeq \pm\frac{1}{2}\ln(4A)$. Note that $\frac{1}{2}\ln(4A) = \frac{1}{2}\ln(16Eg/\sigma) = \frac{1}{2}(\Phi_0 + \Phi_E)$. We remark that, for $A = \frac{4Eg}{\sigma} \gg 1$, we have $\int_0^\infty\mathrm{d}\varphi\,\Upsilon_A(\varphi) \simeq 1$ and $\int_0^\infty\mathrm{d}\varphi\,\varphi\,\Upsilon_A(\varphi) \simeq \frac{1}{2}\ln(4A)$ (these equalities are already excellent for $A = 0.5$). Therefore, using the approximate form of the distribution derived above, we can write

$$\gamma \simeq \frac{2g}{(\Phi_0 - \Phi_E)^2}\left[\int_{-\infty}^0\mathrm{d}\varphi(\varphi + \Phi_0)\Upsilon_A(\varphi) + \int_0^\infty\mathrm{d}\varphi(\varphi - \Phi_E)\Upsilon_A(\varphi)\right] = \frac{2g}{\Phi_0 - \Phi_E}. \quad (81)$$

Therefore, we recover the result obtained for the supersymmetric Hamiltonian alone [15]

$$\boxed{\gamma(E) \simeq \frac{2g}{\ln(16g^2/E)}} \qquad \text{for} \quad \sqrt{g\sigma} \ll E \ll g^2. \quad (82)$$

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High-energy limit. In the high-energy limit the distribution of the phase θ is almost flat, therefore using (77)

$$\gamma(E \to +\infty) \simeq \frac{\sigma}{8E} + \frac{g}{2} = \gamma_{\text{scalar}} + \gamma_{\text{susy}}, \tag{83}$$

where $\gamma_{\text{scalar}} \simeq \frac{\sigma}{8E}$ and $\gamma_{\text{susy}} \simeq \frac{g}{2}$ are the high-energy Lyapunov exponents for $H_{\text{scalar}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ and H_{susy}, respectively.

For $E \to \infty$ the localization length saturates to $\ell_{\text{loc}} \simeq 2/g$. The high-energy wavefunctions present rapid oscillations over a scale $1/k$ exponentially damped on a larger scale $2/g$.

Negative part of the spectrum: $E = -k^2$. As we have seen above, compare to the SDE for $E = +k^2$, the SDE for the variable ξ for $E = -k^2$ receives an additional term $k \sin 2\theta$, therefore

$$\gamma = k \langle \sin 2\theta \rangle - \frac{\sigma}{2k^2} \langle \sin^2 \theta \cos 2\theta \rangle + g \langle \sin^2 2\theta \rangle. \tag{84}$$

In the limit $E \to -\infty$ the phase is trapped at $\theta \simeq \pi/4$ (this is related to trapping of φ by the local minimum of potential $U(\varphi)$ at φ_+), therefore

$$\gamma(E \to -\infty) \simeq \sqrt{-E} + g. \tag{85}$$

This increase of the Lyapunov exponent reflects that the low-energy wavefunctions are sharply peaked around deep wells of the potential.

4. Replica method

In this section, we derive analytic expressions for the IDoS and the Lyapunov exponent by using the replica method. The computation consists of a slight variant of the method used in [15], which leads to hypergeometric functions, generalizing the Bessel and Airy functions appearing in the pure supersymmetric and pure scalar potential problem respectively [57]. Therefore, we only sketch the main lines and refer to [15] for details.

We consider the Hamiltonian (2) with $V(x)$ and $\phi(x)$ two uncorrelated Gaussian white noises, in the more general case where $\langle \phi(x) \rangle$ is finite: $V(x) = \sqrt{\sigma}\eta(x)$ and $\phi(x) = \mu g + \sqrt{g}\tilde{\eta}(x)$ ($\eta(x)$ and $\tilde{\eta}(x)$ with $\mu > 0$ are two uncorrelated normalized Gaussian white noises of zero means). As mentioned above, the problem of δ-correlations between the noises may be mapped on the uncorrelated case (see appendix A). The spectral properties of H are encoded in Green's function $G(x, y; E)$ given by the matrix element

$$G(x, y; E) = (x|\frac{1}{E - H}|y) = \sum_\alpha \frac{\Psi_\alpha(x)\Psi_\alpha^*(y)}{E - E_\alpha} \tag{86}$$

in position space. Here Ψ_α denotes the eigenfunction associated with the energy level E_α, and the sum runs over all states α. According to Thouless' formula, average with respect to disorder $\langle G(x, x; E) \rangle$ of Green's function at equal points yields the derivative of Lyapunov exponent as a function of E [58]. Analytic continuation $E \to E - \mathrm{i}0^+$ allows us to write

$$\langle G(x, x; E - \mathrm{i}0^+) \rangle = \gamma'(E) + \mathrm{i}\pi\rho(E), \tag{87}$$

where $\rho(E)$ is the density of states per unit length.

4.1. The n-replica Hamiltonian

We shall make use of the replica trick in order to compute the averaged equal-point Green's function (87) (see, for example, [39]). To this end, we introduce an auxiliary n-component field $\chi = (\chi^1, \ldots, \chi^n)$ and rewrite $\langle G(x, x; E) \rangle$ in terms of a Gaussian path integral with respect to χ,

$$
\begin{aligned}
\langle (x|(H-E)^{-1}|x) \rangle &= \frac{1}{L} \int_{-L/2}^{+L/2} dx \, \langle (x|(H-E)^{-1}|x) \rangle \\
&= \frac{1}{L} \lim_{n \to 0} \frac{\partial}{\partial n} \int_{-L/2}^{+L/2} dx \int \mathcal{D}\chi \, \chi(x)^2 \left\langle \exp\left(-\frac{1}{2} \int_{-L/2}^{+L/2} dy \, \chi(y)(H-E)\chi(y) \right) \right\rangle \\
&= \frac{2}{L} \frac{\partial}{\partial E} \lim_{n \to 0} \frac{\partial}{\partial n} \int \mathcal{D}\chi \left\langle \exp\left(-\frac{1}{2} \int_{-L/2}^{+L/2} dy \, \chi(y)(H-E)\chi(y) \right) \right\rangle.
\end{aligned}
\tag{88}
$$

Note that the first line makes explicit use of translation invariance after average with respect to disorder. The limit $n \to 0$ eliminates the residual determinant from path integration with respect to χ. We thus are interested in the n-replica partition function

$$
\begin{aligned}
\mathcal{Z}_n &= \int \mathcal{D}\chi \left\langle \exp\left(-\frac{1}{2} \int_{-L/2}^{+L/2} dx \, \chi(x)(H-E)\chi(x) \right) \right\rangle \\
&= \int \mathcal{D}\chi \, \exp\left(-\int_{-L/2}^{+L/2} dx \, \mathcal{L}(\chi, \dot\chi) \right),
\end{aligned}
\tag{89}
$$

where the average over disorder has lead to the Lagrangian

$$
\begin{aligned}
\mathcal{L}(\chi, \dot\chi) = &\frac{1}{2}\dot\chi^2 - \frac{g}{2(1+g\chi^2)}(\chi \cdot \dot\chi)^2 + \frac{\mu^2 g^2 \chi^2}{1+g\chi^2} \\
&- \frac{1}{2}E\chi^2 - \frac{1}{8}\sigma(\chi^2)^2 + \frac{1}{2}\delta^{(n)}(0)\ln\det(1+g\chi^2).
\end{aligned}
\tag{90}
$$

As the formula suggests, we abbreviate $\chi^2 = \sum_i (\chi^i)^2$ and the scalar product $\chi \cdot \eta = \sum_i \chi^i \eta^i$. Rewriting $\mathcal{L}$ as

$$
\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij}(\chi) \dot\chi^i \dot\chi^j + V(\chi), \qquad \eta_{ij}(\chi) = \delta_{ij} - \frac{g\chi_i \chi_j}{1+g\chi^2}
\tag{91}
$$

shows that the Lagrangian describes the motion of a point particle in an n-dimensional curved space with metric η_{ij}. The potential is given by

$$
V(\chi) = \frac{\mu^2 g^2 \chi^2}{1+g\chi^2} - \frac{1}{2}E\chi^2 - \frac{1}{8}\sigma(\chi^2)^2 + \frac{1}{2}\delta^{(n)}(0)\ln\det(1+g\chi^2).
\tag{92}
$$

The contact term $\delta^{(n)}(0)$ may be eliminated by introducing an auxiliary field $\Sigma = \sqrt{1+g\chi^2}$ and rewriting the functional integration measure as $\mathcal{D}\chi \mathcal{D}\Sigma \delta(\Sigma^2 - g\chi^2 - 1)$; following [15] this term will not be considered in what follows. We recognize an σ-model with symmetry group $O(n, 1)$. In one spatial dimension, we may transform it to a quantum-mechanical problem in n-dimensions where x plays the role of time. Hence we must identify a proper Hamiltonian $\mathcal{H}$ related to $\mathcal{L}$ and study its spectrum. $\mathcal{H}$ acts on a Hilbert space with inner product

$$
(\Phi, \Psi) = \int_{\mathbb{R}^n} d^n\chi \sqrt{\det\eta} \, \Phi^*(\chi)\Psi(\chi) = \int_{\mathbb{R}^n} \frac{d^n\chi}{\sqrt{1+g\chi^2}} \Phi^*(\chi)\Psi(\chi)
\tag{93}
$$

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and its eigenvalues $\mathcal{E}_\nu(n)$ and eigenfunctions $\Psi_\nu(\chi)$ are given by the solutions of $\mathcal{H}\Psi_\nu(\chi) = \mathcal{E}_\nu(n)\Psi_\nu(\chi)$ with $\|\Psi_\nu\|^2 = (\Psi_\nu, \Psi_\nu) < \infty$. Since the derivation of $\mathcal{H}$ is very much like in [15] we only state the result. From (91) and (92) we find the n-replica Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(-\Delta + (1-n)\chi \cdot \nabla - (\chi \cdot \nabla)^2 + \frac{n}{2} - \frac{1}{4}\frac{\chi^2}{1+\chi^2}\right) + V(\chi). \tag{94}$$

The angular eigenstates are given by the Gegenbauer polynomials $C_\ell^{n/2-1}(\cos\theta)$ where ℓ denotes the main angular quantum number. After separation of the angular part, we are left with the radial part of the Hamiltonian that depends only on the modulus $\rho = \sqrt{\chi^2}$,

$$\mathcal{H}_\mathrm{r} = -\frac{1}{2}(1+g\rho^2)\frac{\partial^2}{\partial\rho^2} - \frac{n-1}{2\rho}\frac{\partial}{\partial\rho} - \frac{ng\rho}{2}\frac{\partial}{\partial\rho} + \frac{\ell(\ell+n-2)}{2\rho^2}$$
$$+ \frac{gn}{4} + \frac{\mu^2 g^2 \rho^2}{1+g\rho^2} - \frac{1}{2}E\rho^2 - \frac{1}{8}\sigma\rho^4. \tag{95}$$

4.2. The ground state: Lyapunov exponent and IDoS

In the limit $L \to \infty$ we expect that the path integral (89) has a leading term $\exp[-L\mathcal{E}_\mathrm{G}(n)/2]$ where $\mathcal{E}_\mathrm{G}(n)/2$ corresponds to the ground-state energy of the Hamiltonian $\mathcal{H}$. Combining (87)–(89) we conclude that

$$\gamma(E) + \mathrm{i}\pi N(E) + \mathrm{const} = -\frac{2}{L}\left.\frac{\partial \mathcal{Z}_n}{\partial n}\right|_{n=0} = \left.\frac{\partial\mathcal{E}_\mathrm{G}(n)}{\partial n}\right|_{n=0}. \tag{96}$$

As above, $N(E)$ denotes the integrated density of states per unit length, and $\gamma(E)$ the Lyapunov exponent. The constant must be chosen in order to ensure correct asymptotic behavior $E \to \pm\infty$ (in particular $N(E \to -\infty) = 0$). We shall discuss this problem below. Following the spirit of the replica method, we analytically continue $\mathcal{E}_\mathrm{G}(n) = n\mathcal{E}_0 + n^2\mathcal{E}_1 + \cdots$ and thus identify

$$\gamma(E) + \mathrm{i}\pi N(E) = \mathcal{E}_0 + \mathrm{const}. \tag{97}$$

We now compute $\mathcal{E}_0$ for the Hamiltonian (95). We expect the ground state to be an s-wave state with total angular momentum $\ell = 0$. Changing variables to $\xi^2 = 1 + g\rho^2$ in (95) leads to the Hamiltonian

$$\mathcal{H}_\mathrm{r} = \frac{g(\xi^2-1)}{2}\mathcal{T} + \frac{ng}{2}\left(\frac{1}{2} - \xi\frac{\partial}{\partial\xi}\right), \qquad \text{with}$$
$$\mathcal{T} = -\frac{\partial^2}{\partial\xi^2} - \frac{E}{g^2} - \frac{\sigma}{4g^3}(\xi^2-1) + \frac{\mu^2-1/4}{\xi^2}. \tag{98}$$

Consequently, we must solve the equation $\mathcal{H}_\mathrm{r}\Psi = \frac{1}{2}\mathcal{E}_\mathrm{G}(n)\Psi$ for the ground-state wavefunction. As for the eigenvalue $\mathcal{E}_\mathrm{G}(n)$, we expand the ground-state wavefunction into a power series with respect to n: $\Psi = \Psi_0 + n\Psi_1 + \cdots$. This yields an infinite system of coupled differential equations whose first two members are

$$\mathcal{T}\Psi_0 = 0 \qquad \text{and} \qquad \frac{g(\xi^2-1)}{2}\mathcal{T}\Psi_1(\xi) + \frac{g}{2}\left(\frac{1}{2} - \xi\frac{\partial}{\partial\xi}\right)\Psi_0(\xi) = \frac{\mathcal{E}_0}{2}\Psi_0(\xi). \tag{99}$$

Since we seek for a normalizable ground-state wavefunction in the limit $n \to 0$ we have to find a square-integrable solution of $\mathcal{T}\Psi_0(\xi) = 0$. Applying the limit $\xi \to 1$ in (99), we finally may relate Ψ_0 to the eigenvalue

$$\mathcal{E}_0 = g\left(\frac{1}{2} - \frac{\xi}{\Psi_0(\xi)}\frac{\partial\Psi_0(\xi)}{\partial\xi}\right)\bigg|_{\xi=1}. \tag{100}$$

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The solution is given in appendix D and yields the wavefunction Ψ_0,

$$\Psi_0(\xi) = \exp\left(-\frac{i\xi^2}{4}\sqrt{\frac{\sigma}{g^3}}\right)\xi^{\mu+1/2}\, U\left(\frac{\mu+1}{2} + \frac{i}{2}\left(\frac{E}{\sqrt{\sigma g}} - \frac{1}{4}\sqrt{\frac{\sigma}{g^3}}\right), \mu+1, \frac{i\xi^2}{2}\sqrt{\frac{\sigma}{g^3}}\right),$$

(101)

where $U(a, b, z)$ denotes the second confluent hypergeometric function [59]. Therefore, $\mathcal{E}_0$ takes the value

$$\mathcal{E}_0 = -\mu g - \frac{i}{2}\sqrt{\frac{\sigma}{g}}\left(1 - \frac{2aU(a+1, b+1, i\sqrt{\sigma/4g^3})}{bU(a, b, i\sqrt{\sigma/4g^3})}\right),$$

(102)

where we have introduced

$$a = \frac{\mu+1}{2} + \frac{i}{2}\left(\frac{E}{\sqrt{\sigma g}} - \frac{1}{4}\sqrt{\frac{\sigma}{g^3}}\right) \qquad \text{and} \qquad b = \mu+1\,.$$

(103)

The imaginary part can be extracted by using the Wronskian (D.5) of $\Psi_0(\xi)$ and its complex conjugate,

$$N(E) = \frac{g}{\pi}\left(\frac{4g^3}{\sigma}\right)^{\frac{\mu}{2}}\frac{\exp(\pi\,\text{Im}\,a)}{|U(a, b, i\sqrt{\sigma/4g^3})|^2}.$$

(104)

We have obtained a compact expression that can be used more conveniently than the double integral (31) in order to plot the IDoS.

Equations (102)–(104) provide an exact solutions for the Lyapunov exponent $\gamma(E)$ as well as the IDoS $N(E)$ for this model, up to a constant which depends upon σ, g and μ, and may be fixed by imposing correct asymptotic behaviors, like $\lim_{E\to-\infty} N(E) = 0$. These results interpolate between the known cases of white noise potential and the random supersymmetric Hamiltonian.

Let us give an example on how to use (102), (103) to study the behavior at $E = 0$ for the Sinai case $\mu = 0$. We have

$$a = \frac{1}{2} - \frac{i}{8}\sqrt{\frac{\sigma}{g^3}} \qquad \text{and} \qquad b = 1.$$

(105)

Recall that the confluent hypergeometric function $U(a, b, z)$ behaves like

$$U(a, b, z) \underset{z\to 0}{\sim} \begin{cases} \Gamma(b-1)/\Gamma(a)z^{1-b}, & b > 1 \\ (\ln z + \psi(a))/\Gamma(a), & b = 1 \end{cases}.$$

(106)

For small Gaussian noise $\sigma \to 0^+$ we tacitly neglect the small imaginary part of a, leading to further corrections, and find

$$\mathcal{E}_0 \underset{E=0}{\approx} +\frac{i}{2}\sqrt{\frac{\sigma}{g}}\left(1 - \frac{2}{\ln\left(\frac{i}{2}\sqrt{\frac{\sigma}{g^3}}\right) + \psi(1/2)}\left(\frac{i}{2}\sqrt{\frac{\sigma}{g^3}}\right)^{-1}\right)$$

(107)

$$\approx -\frac{2g}{\left(\ln\left(\frac{1}{2}\sqrt{\frac{\sigma}{g^3}}\right) + \psi(1/2)\right)^2 + \frac{\pi^2}{4}}\left(\ln\left(\frac{1}{2}\sqrt{\frac{\sigma}{g^3}}\right) + \psi(1/2) - \frac{i\pi}{2}\right).$$

(108)

Therefore we obtain the approximate IDoS,

$$N(0) \simeq \frac{g}{[\ln\sqrt{g^3/\sigma} + \ln 2 - \psi(1/2)]^2 + \pi^2/4}.$$
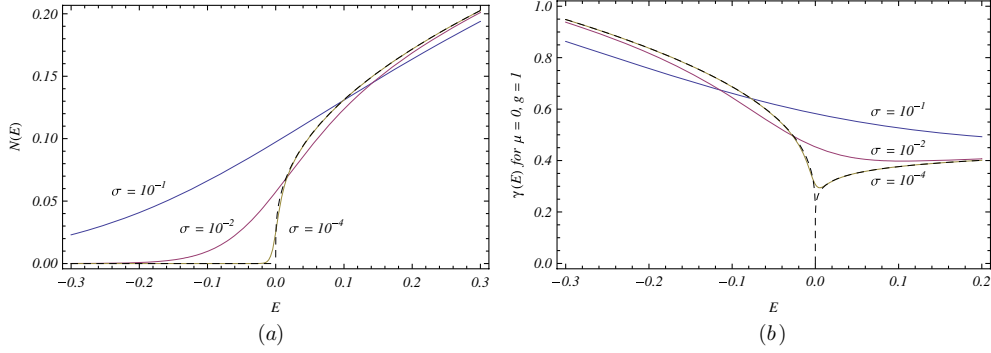
(109)

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Figure 6. IDoS (left) and Lyapunov exponent (right) for $g = 1$ and $\mu = 0$ for various values of σ. Delocalization transition at $E = 0$ for $\sigma = 0$ (dashed lines) is suppressed even by a tiny scalar potential.
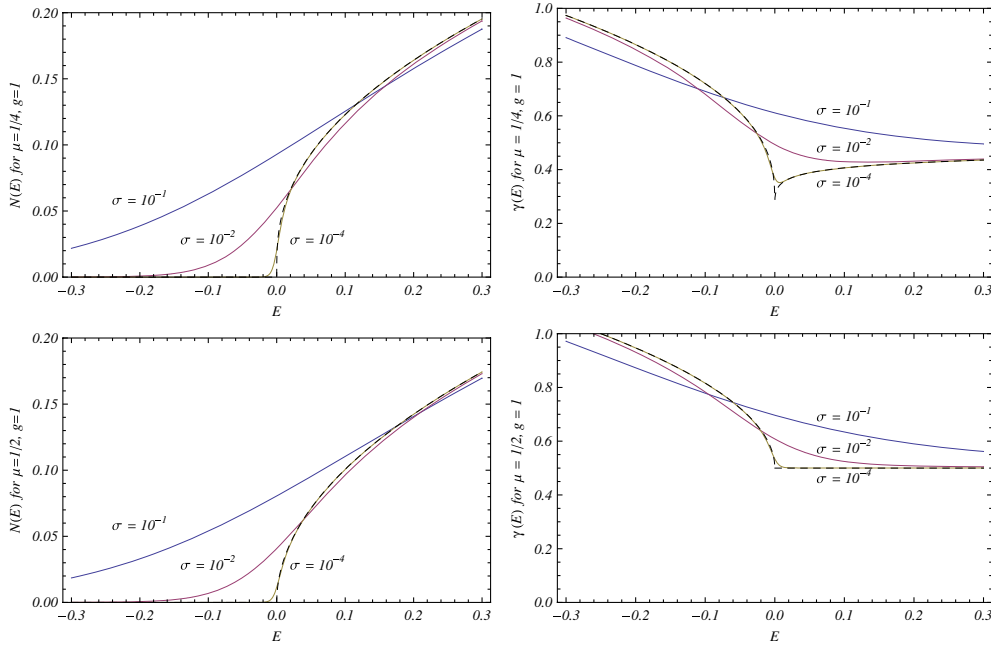

Figure 7. $N(E)$ and $\gamma(E)$ for $\mu = 1/4$ (top) and $\mu = 1/2$ (bottom). The dashed lines correspond to the pure supersymmetric results for $\sigma = 0$.

We have recovered by the replica method the behavior obtained in sections 2 and 3,

$$N(E = 0) \underset{\sigma \to 0}{\sim} \frac{g}{\ln^2(g^3/\sigma)} \qquad \text{and} \qquad \gamma(E = 0) \underset{\sigma \to 0}{\sim} \frac{g}{\ln(g^3/\sigma)}. \qquad (110)$$

Note however that the next leading order are different (this is not surprising since the approximation scheme of section 2 is quite different). Nevertheless, (102)–(104) are less manageable for the intermediate regimes singled out in the previous sections.

Figure 6 illustrates $N(E)$ and $\gamma(E)$ for the Sinai case ($\mu = 0$, $g = 1$). Any Gaussian noise with $\sigma > 0$ lifts the singular behavior $N_{\text{susy}}(E) \sim 1/(\ln E)^2$ and $\gamma_{\text{susy}}(E) \sim 1/|\ln E|$ to

analyticity in the vicinity of $E = 0$. In particular, as shown in figure 6, any small σ shifts the singularity of $\gamma(E)$ to some minimum at some $E_{\min} > 0$.

The case $\mu \neq 0$. It is also interesting to consider the case of finite $\langle\phi\rangle = \mu g$. In the absence of the scalar noise $V(x)$ ($\sigma = 0$) the power-law Dyson singularity of the IDoS is transformed into a power-law behavior $N(E) \sim E^{\mu}$. If a tiny scalar noise is introduced a fraction of states migrates to $\mathbb{R}^-$,

$$N(E = 0) \underset{\sigma \to 0}{\simeq} \frac{g}{\pi} \left[\frac{\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma(\mu)} \right]^2 \left(\frac{\sigma}{4g^3} \right)^{\frac{\mu}{2}}, \tag{111}$$

which we find by straightforward application of (106)–(104). Moreover, the feature of smoothing singular behavior extends to $0 < \mu < 1/2$. For $\sigma = 0$ we have the non-analytic behavior $\gamma_{\text{susy}}(E) \sim \mu g + C_{\pm}|E|^{\mu}$ with some constants $C_{\pm}$ for $E > 0$ and $E < 0$ respectively. Again, the introduction of σ shifts this power-law singularity to some minimum of $\gamma(E)$ at small positive $E_{\min}$, as illustrated on figure 7. In either case, the evaluation of $E_{\min}$ seems to be difficult. However, it would be interesting to find a physical argument for this mechanism.

5. Conclusion

In this paper, we have studied the spectral and localization properties of a one-dimensional random Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x) + V(x) = H_{\text{susy}} + V(x)$ which interpolates between the well-studied examples of random supersymmetric models H_{susy} and Halperin's model H_{scalar}. Our analysis has pointed out a natural competition between the fluctuations of $\phi(x)$ and $V(x)$. We have identified the important scales that control this competition for $g^3 \gg \sigma$ or $g^3 \ll \sigma$, which are the two largest scales among σ/g, $\sigma^{2/3}$, $\sqrt{g\sigma}$ and g^2. We have observed that even a small additional scalar noise $V(x)$ lifts the singular spectral and localization properties of H_{susy}: the Dyson singularity of the IDoS and the vanishing of the Lyapunov exponent at $E = 0$ are replaced by smooth behaviors: a small additional scalar white noise ($\sigma \to 0$) leads to a migration of a fraction $N(0) \sim g/\ln^2(g^3/\sigma)$ of eigenstates to negative values. It is worth noting that, $\forall \mu$, the zero energy IDoS (110), (111) for $g^3 \gg \sigma$ can be obtained by the substitution $E \to \sqrt{g\sigma}$ in the known expressions for $\sigma = 0$,

$$N^{(\sigma \neq 0)}(E = 0) \sim N^{(\sigma=0)}(E \sim \sqrt{g\sigma}). \tag{112}$$

This is a simple consequence of the correct identification of the crossover energy scales.

Simultaneously to the smoothing of the Dyson singularity, the delocalization transition of H_{susy} at $E = 0$ disappears and the Lyapunov exponent takes a finite value $\gamma(0) \sim g/\ln(g^3/\sigma)$. This logarithmic behavior shows that, in practice (see figure 6), even a tiny Gaussian noise σ kills the singularity of the Lyapunov that becomes almost flat $\gamma(E) \sim g$ for all energies for which density of states is significant.

IDoS and Lyapunov exponent have also been studied in the other regimes. In particular, how the fraction $N(0) \sim g/\ln^2(g^3/\sigma)$ of states are distributed among negative energies has been further analyzed; the precise (Lifshits) exponential tail of the IDoS has been derived in the various regimes. It is worth emphasizing that in the lowest part of the spectrum, the tail involves a competition between the supersymmetric and the scalar noise, $N(E \to -\infty) \sim \exp\left(-\frac{\pi}{\sqrt{g\sigma}}|E|\right)$, whatever is the largest scale among g^3 (supersymmetric noise) and σ (scalar noise).

The study of spectral properties has been completed by considering the individual distributions of eigenenergies (extreme value problem). We have shown that these distributions

coincide with Gumbel laws, a consequence of the absence of spectral correlations due to the strong localization of the wavefunctions [48], like for the scalar potential alone; this can be opposed to the purely supersymmetric case ($\sigma = 0$) for which distributions of eigenenergies are strongly modified in the neighborhood of the delocalization transition [33].

The study of individual distributions of eigenenergies, that includes properly finite size (Dirichlet boundary) effects, had allowed us to identify the critical value σ_c of the scalar noise σ below which, for fixed g and $L \gg 1/g$, the scalar noise can be ignored. We have obtained $\sigma_c \sim g^3 \, e^{-\sqrt{gL}}$. It is worth noting that the corresponding value of the $E = 0$ Lyapunov exponent (roughly its minimum value) then reads $\gamma(0) \sim g/\ln(g^3/\sigma_c) \sim \sqrt{g/L}$. This corresponds to a maximum localization length $\ell_{\mathrm{loc}} \sim \sqrt{L/g} \ll L$.

It is not too surprising that the additional white noise modifies spectral and localization properties in the vicinity of the band center (around $E = 0$). However, it is somewhat unexpected that, at any value of g (even in the limit $g \to 0^+$), the noise $\phi(x)$ from the supersymmetric part controls the spectral properties for $E \to -\infty$, which we have seen on the tail $N(E) \sim \exp\left(-\frac{\pi}{\sqrt{g\sigma}}|E|\right)$ and the distributions of the lowest energy levels. This feature seems counter-intuitive since the pure SUSY spectrum is strictly positive so that we would have expected the potential $V(x)$ to yield the behavior $N(E) \sim \exp\left(-\frac{8}{3\sigma}|E|^{3/2}\right)$. We attribute this behavior to the singular nature of the supersymmetric potential $\phi(x)^2 + \phi'(x)$ which is also responsible for the saturation of the Lyapunov exponent at high energies $\gamma(E) \simeq g/2$ for $E \to +\infty$. Part of this picture will change if supersymmetric noise is replaced by a more regular process with regular correlation function of finite width and height (see [34]).

Diffusion in a random force field with random annihilation/creation rates. Finally, it is interesting to come back to the analysis of the results in the context of classical diffusion in a random force field with random annihilation/creation rates. In order to distinguish more clearly the roles of the force field $\phi(x)$ and the annihilation/creation rates $V(x)$, we consider several situations and analyze the density of particles $\langle n(x, t|x, 0)\rangle$ at x at time t, when a particle has been released at x initially. Averaging is taken over the random force field and the random annihilation/creation rates.

- *For $g = 0$ and $\sigma = 0$:* it is useful to recall the obvious fact that in the absence of random force field and absorption we have $n(x, t|x, 0) = \frac{1}{\sqrt{4\pi t}}$.
- *For $g \neq 0$ and $\sigma = 0$:* classical diffusion in a random force field (Sinai problem). Thanks to (9), the spectral Dyson singularity $N(E) \sim 1/\ln^2 E$ can be connected to large time behavior [15]

$$\langle n(x, t|x, 0)\rangle \underset{t\to\infty}{\sim} \frac{1}{\ln^2 t} \tag{113}$$

much slower than the $1/\sqrt{t}$. This behavior is related to the behavior $x(t) \sim \ln^2 t$ of the typical distance covered by the random walker [15] (see also [18] where many interesting properties of the Sinai problem were studied thanks to the powerful real space renormalization group method of Ma and Dasgupta).

- *For $g = 0$ and $\sigma \neq 0$:* in order to examine the effect of the annihilation/creation rates that were chosen to be zero on average, we first switch off the random force field. Of course the number of particles is not conserved for $\sigma \neq 0$. In this case, the spectral Lifshits singularity of the DoS is $\rho(E) \simeq \frac{1}{2\pi\sqrt{|E|}} \exp{-\frac{8|E|^{3/2}}{3\sigma}}$. The Laplace transform (9) is dominated by negative energy contributions. A steepest descent estimation shows that the averaged number of returning particles diverges with time as

$$\langle n(x, t|x, 0)\rangle \underset{t\to\infty}{\simeq} \frac{1}{\sqrt{\pi t}} \, e^{+\frac{\sigma^2}{48}t^3}. \tag{114}$$

We emphasize that this increase of the averaged density cannot be compensated by a finite mean value of the annihilation rates $\langle V \rangle > 0$ that would only add a $\mathrm{e}^{-\langle V \rangle t}$ to this result.

- *For $g \neq 0$ and $\sigma \neq 0$*: finally, we consider the case of a random force field with random annihilation/creation rates. The form taken by the Lifshits singularity $\rho(E) \sim \exp -\frac{\pi |E|}{\sqrt{\sigma g}}$ leads to the surprising conclusion that the average number of returning particles diverges at a finite time $t_c = \pi/\sqrt{g\sigma}$,

$$\langle n(x, t | x, 0) \rangle = \infty \qquad \text{for} \quad t \geqslant t_c. \tag{115}$$

The two previous points show that this divergence of the *average* particle density is due to the interplay between the random force field and the random annihilation/creation rates. It would be an interesting issue to understand precisely the physical origin of this remark. On the other hand, these last remarks might indicate that the white noise $V(x)$ is probably too widely fluctuating for a reasonable description of a realistic random annihilating/creating rates. Maybe a more interesting model would be to add a low concentration of such sites. In the continuum limit this would correspond to add to the supersymmetric Hamiltonian a scalar potential of the form $V(x) = \sum_n \alpha_n \delta(x - x_n)$, where x_n are random positions with a density ρ and α_n local annihilation/creation rates. The limit of high density $\rho \gg |\alpha_n|$ corresponds to the white noise limit studied in the present paper. The limit of low density $\rho \ll |\alpha_n|$ might be more interesting. This model has been recently studied in the absence of the random force field and for absorbing sites ($\alpha_n > 0$) in [60], where a penetration length was derived in any dimension thanks to renormalization group methods. An interesting question would be to understand the effect of the random force field on these known properties.

Acknowledgments

Appendix A. The case of correlated noises

It is possible to extent the analysis to correlated noises in the following sense. Suppose that $V(x) = \sqrt{\sigma}\eta(x)$ and $\phi(x) = \mu g + \sqrt{g}\tilde{\eta}(x)$ are correlated such that

$$\langle V(x)\phi(y) \rangle = \Gamma \delta(x - y). \tag{A.1}$$

We introduce variables $\zeta(x) = \phi(x) - A$ and $v(x) = 2A(\phi(x) - \mu g) + V(x)$ so that the Hamiltonian may be rewritten as

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \zeta(x)^2 + \zeta'(x) + v(x) + 2\mu g A - A^2. \tag{A.2}$$

The new variables have the correlation function

$$\langle \zeta(x)v(y) \rangle = (2gA + \Gamma)\delta(x - y), \tag{A.3}$$

so that the choice $A = -\Gamma/2g$ makes them independent. Further characteristics are given by

$$
\begin{aligned}
\langle v(x) \rangle = 0, \qquad & \langle v(x)v(y) \rangle = \sigma \delta(x - y) \\
\langle \zeta(x) \rangle = \mu g + \frac{\Gamma}{2g}, \qquad & \langle \hat{\zeta}(x)\hat{\zeta}(y) \rangle = g \delta(x - y),
\end{aligned}
\tag{A.4}
$$

where $\hat{\zeta}(x) = \zeta(x) - \mu g - \Gamma/2g$. Thus, up to a re-definition of energy $\varepsilon = E + \mu\Gamma + \Gamma^2/4g^2$, we recover the problem of uncorrelated noises.

Appendix B. A useful relation

Let us consider a random process generated by uncorrelated Wiener processes $dW_i(t)$,

$$\boxed{dx(t) = a(x)\,dt + b_i(x)\,dW_i(t) \overset{\text{(law)}}{=} a(x)\,dt + \sqrt{b_i(x)b_i(x)}\,dW(t)}. \qquad \text{(B.1)}$$

The equality is valid for Ito and Stratonovich prescriptions. Let us demonstrate this relation.

Ito's prescription. Recall that the SDE

$$dx_i = a_i(x)\,dt + b_{ij}(x)\,dW_j(t) \qquad \text{(Ito)} \qquad \text{(B.2)}$$

is associated with a FPE $\partial_t P = F_x P$ where the Forward Fokker–Planck generator is [38]

$$F_x = -\partial_i a_i + \tfrac{1}{2}\partial_i \partial_j b_{ik}b_{jk}. \qquad \text{(B.3)}$$

Therefore $dx_i = a(x)\,dt + b_j(x)\,dW_j$ is associated with a FPE with generator $F_x = -\partial_x a(x) + \tfrac{1}{2}\partial_x^2 b_j(x)b_j(x)$ that is also associated with the SDE $dx = a(x)\,dt + \sqrt{b_i(x)b_i(x)}\,dW(t)$.

Stratonovich's prescription. The relation between Ito and Stratonovich prescriptions is given in [38]

$$dx = \alpha(x)\,dt + \beta_j(x)\,dW_j(t) \qquad \text{(Stratonovich)} \qquad \text{(B.4)}$$

$$= \left[\alpha + \tfrac{1}{2}\beta_j\beta_j'\right]dt + \beta_j\,dW_j(t) \qquad \text{(Ito)} \qquad \text{(B.5)}$$

$$\overset{\text{(law)}}{=} \left[\alpha + \tfrac{1}{2}\beta_j\beta_j'\right]dt + \sqrt{\beta_j\beta_j}\,dW(t) \qquad \text{(Ito)} \qquad \text{(B.6)}$$

$$= \left[\alpha + \tfrac{1}{2}\beta_j\beta_j' - \tfrac{1}{2}\sqrt{\beta_j\beta_j}(\sqrt{\beta_j\beta_j})'\right]dt + \sqrt{\beta_j\beta_j}\,dW(t) \qquad \text{(Stratonovich)} \qquad \text{(B.7)}$$

$$= \alpha(x)\,dt + \sqrt{\beta_j(x)\beta_j(x)}\,dW(t) \qquad \text{(Stratonovich)}. \qquad \text{(B.8)}$$

This shows that addition law of variances holds not only for additive processes but also for multiplicative processes.

Appendix C. A remark on the Lyapunov exponent

In this appendix, we clarify some relations between different formulae for the Lyapunov exponent given above.

Let us present the problem with the well-known Halperin model $H_{\text{scalar}} = -\frac{d^2}{dx^2} + V(x)$. Here $V(x)$ denotes a white-noise potential with average $\langle V(x) \rangle = 0$, and $\langle V(x)V(y) \rangle = \sigma\delta(x-y)$. The widely-used Ricatti mapping allows us to relate the spectral statistics for H_{scalar} to passage probabilities for a diffusion $z(x)$ whose evolution is governed by the SDE $z'(x) = -[E + z(x)^2] + V(x)$. In particular, the stationary distribution $T(z)$ for z is solution to the differential equation,

$$\frac{\sigma}{2}T'(z) + (z^2 + E)T(z) = N(E), \qquad \text{(C.1)}$$

where $N(E)$ denotes the integrated density of states for H_{scalar} as it can be shown from the node-counting theorem. Moreover, the Lyapunov exponent relates to the diffusion via Rice formula $\gamma = \langle z \rangle$ which, however, must be understood as the principal value

$$\gamma = \lim_{R \to +\infty} \int_{-R}^{R} dz\, z T(z) \qquad \text{(C.2)}$$

in order to avoid difficulties from the asymptotic behavior $T(z) \sim N(E)/z^2$ as $|z| \to +\infty$. For $E > 0$ (C.1) allows us to rewrite

$$\gamma = \lim_{R \to \infty} \int_{-R}^{R} \mathrm{d}z \, z \left(\frac{N(E)}{z^2 + E} - \frac{\sigma}{2(z^2 + E)} T'(z) \right). \tag{C.3}$$

Clearly, the first term yields 0. Note that it is crucial to let the integration bounds tend to 0 symmetrically, otherwise we would not find a well-defined result. After partial integration of the second term, we eventually find an alternative expression for the Lyapunov exponent

$$\gamma(E > 0) = -\frac{\sigma}{2} \int_{-\infty}^{+\infty} \mathrm{d}z \, \frac{z^2 - E}{(z^2 + E)^2} T(z). = -\frac{\sigma}{2} \left\langle \frac{z^2 - E}{(z^2 + E)^2} \right\rangle. \tag{C.4}$$

The integration does not require anymore the principal value: it was possible to let the cutoff R go to infinity since integrand now vanishes sufficiently fast thanks to the partial integration.

This relation is particularly useful in order to study the limit $E \to \infty$ since we may immediately read of the asymptotic behavior $\gamma \propto \sigma/E$. However the drawback is that (C.4) is rather ill-defined for $E \leqslant 0$. Nevertheless, writing

$$T(z) = \frac{N(E)}{z^2 - E} - \frac{2ET(z)}{z^2 - E} - \frac{\sigma T'(z)}{2(z^2 - E)}, \tag{C.5}$$

it is not difficult to show that

$$\gamma(E < 0) = -\left\langle \frac{2Ez}{z^2 - E} \right\rangle - \frac{\sigma}{2} \left\langle \frac{z^2 + E}{(z^2 - E)^2} \right\rangle \tag{C.6}$$

by partial integration. In order to extract the asymptotics, recall that as $E \to -\infty$ the distribution $T(z)$ is centered at $z \sim \sqrt{-E}$. Using this scaling behavior we recover the asymptotic behavior $\gamma \propto \sqrt{-E}$. It remains that the $E \to 0$ limit in the two relations (C.4), (C.6) seems tricky.

Let us now turn to our model Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x) + V(x)$. Section 2.1 provides a detailed account on the Ricatti mapping in this case, in particular the stationary distribution $T(z)$ of the variable $z(x)$ was shown to be a solution of the differential equation $N(E) = (z^2 + E + 2gz)T(z) + (\sigma + 4gz^2)T'(z)/2$, see (12). For $E > 0$ we may rewrite

$$T(z) = \frac{N(E)}{z^2 + E} - \frac{2gzT(z)}{z^2 + E} - \frac{(\sigma + 4gz^2)T'(z)}{2(z^2 + E)} \tag{C.7}$$

and insert this expression into (C.2) what indeed allows us to recover (78),

$$\gamma(E > 0) = \frac{\sigma}{2} \left\langle \frac{E + \left(\frac{8Eg}{\sigma} - 1 \right)z^2}{(E + z^2)^2} \right\rangle. \tag{C.8}$$

Conversely, for $E < 0$ we may follow the same strategy as for Halperin's model what yields an additional term

$$\gamma(E < 0) = -\left\langle \frac{2Ez}{z^2 - E} \right\rangle - \frac{\sigma}{2} \left\langle \frac{E + \left(\frac{8Eg}{\sigma} + 1 \right)z^2}{(z^2 - E)^2} \right\rangle. \tag{C.9}$$

Again, the advantage of these formulae is that they provide the asymptotic behavior of γ as $E \to \pm\infty$ in a very explicit way. For example, as $E \to +\infty$ (C.8) shows that $\gamma \propto \sigma/E + 4g$ what is coherent with $\gamma \sim \gamma_{\text{susy}} + \gamma_{\text{scalar}}$.

Appendix D. Solution of the differential equation $\mathcal{T}\Psi_0 = 0$

The differential equation for Ψ_0 is given by

$$\left(-\frac{\partial^2}{\partial\xi^2} - \frac{E}{g^2} - \frac{\sigma}{4g^3}(\xi^2 - 1) + \frac{\mu^2 - 1/4}{\xi^2}\right)\Psi_0(\xi) = 0. \tag{D.1}$$

In the absence of diagonal Gaussian disorder $\sigma = 0$ we recover the solution given in [15]. For $\sigma > 0$, we convert the preceding equation into a differential equation for confluent hypergeometric functions. Indeed, the ansatz

$$\Psi_0(\xi) = \exp(\lambda\xi^2/2)\xi^\alpha w(z) \qquad \text{with} \quad z = \eta\xi^2/2, \tag{D.2}$$

where $\alpha = 1/2 \pm \mu$, $\eta = -2\lambda = \pm i\sqrt{\sigma/g^3}$ leads to

$$w''(z) + (b - z)w'(z) - aw(z) = 0 \qquad \text{with} \quad b = \frac{1}{2} + \alpha, \quad a = \frac{b}{2} - \frac{1}{2\eta}\left(\frac{E}{g^2} - \frac{\sigma}{4g^3}\right). \tag{D.3}$$

The choice $\alpha = 1/2 + \mu$, $\eta = -2\lambda = +i\sqrt{\sigma/g^3}$ leads to a square integrable solution

$$\Psi_0(\xi) = \exp\left(-\frac{i}{4}\sqrt{\frac{\sigma}{g^3}}\xi^2\right)\xi^{\mu+1/2}U\left(\frac{\mu+1}{2} + \frac{i}{2}\left(\frac{E}{\sqrt{g\sigma}} - \frac{1}{4}\sqrt{\frac{\sigma}{g^3}}\right), \mu+1, i\sqrt{\frac{\sigma}{4g^3}}\xi^2\right). \tag{D.4}$$

In order to see square integrability, recall that E has a small negative imaginary part $-i\epsilon$. Using $U(a, b; z) \underset{z\to\infty}{\sim} z^{-a} + \cdots$ we find $|\Psi(\xi)|^2 \underset{\xi\to\infty}{\sim} \xi^{-1-\epsilon}$. A second, linearly independent solution is readily found from the complex conjugation of (D.4). The Wronskian which turns out to be useful for the determination of the integrated density of states may be obtained from known properties of $U(a, b, z)$,

$$W(\Psi_0(\xi), \overline{\Psi_0}(\xi)) = 2i\exp(\pi\,\text{Im}\,a)\left(\frac{4g^3}{\sigma}\right)^{\mu/2}\xi^{1/2-\mu}\exp\left(\frac{i}{4}\sqrt{\frac{\sigma}{g^3}}\xi^2\right). \tag{D.5}$$

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